PAN
ARCHIVES OF ACOUSTICS
Vol. **43**, No. 1, pp. 11–20 (2018)

DOI: 10.24425/118076

Research Papers

Dispersion of Rayleigh Waves in a Microstructural Couple Stress Substrate Loaded with Liquid Layer Under the Effects of Gravity

Vikas SHARMA[(1)], Satish KUMAR[(2)]

[(1)] *Department of Mathematics*
Lovely Professional University
Phagwara, Punjab, 144411, India; e-mail: vikassharma10a@yahoo.co.in, vikas.sharma@lpu.co.in

[(2)] *School of Mathematics*
Thapar University
Patiala, Punjab, 147004, India; e-mail: satishk.sharma@thapar.edu

(*received July 13, 2016; accepted August 28, 2017*)

Bone loss is one of the serious health issues in bedridden patients or young generation due to lack of physical activities. Mechanical forces are exerted on the bones through ground reaction forces, liquid loadings and by other contraction activities of the muscles. We are assuming an isotropic half-space with mechanical properties equivalent to that of bone exhibiting microstructures. Consistent couple stress theory introduces an additional material parameter called characteristic length which accounts for inner microstructure of the material. Dispersion relations for leaky Rayleigh waves are derived by considering a model consisting of couple stress half space under the effects of gravity and loaded with inviscid liquid layer of finite thickness or a liquid half space. Impact of the gravity, liquid loadings and microstructures of the material are investigated on propagation of leaky Rayleigh type waves. Phase velocity of leaky Rayleigh waves is studied for five different values of characteristic length parameter which are of the order of internal cell size of the considered material. Variations in phase velocity of leaky Rayleigh waves are also studied under the effect of gravity parameter and thickness of liquid loadings.

Keywords: couple stress theory; Rayleigh waves; gravity; characteristic length; liquid loading.

1. Introduction

Ultrasonic wave propagation in cortical bone is one of the viable means to study the properties of bones (Vavva *et al.*, 2014). These methods employ classical theory of elasticity and cannot explain the dispersive nature of Rayleigh, guided waves because they lack length scale parameters to incorporate the microstructural effects. Classical theory was based on the assumptions that all materials are homogeneous and are continuously distributed over its volume without any kind of defects, so that even the smallest particle of the material possesses same physical properties as the whole material. It is also assumed that a point particle is represented by a geometrical point which is infinitesimal in size without any internal structure. There are many materials like particulate composites, polymers, cellular solids and cortical bones which do not satisfy above said assumptions. These short comings of classical theory has led to the development of new micromechanical theories of solids which accounts for microstructure of the material and are called microcontinuum theories (Voigt, 1887; Cosserat, Cosserat, 1909; Toupin, 1962; Mindlin, Tiersten, 1962; Koiter, 1964; Eringen, 1968; Nowacki, 1974, Yang *et al.*, 2002).

One common characteristic of these microcontinuum theories was the introduction of couple stresses in addition to force stress tensor. In these theories, the concept of couple stress was introduced by defining the deformation of the material through displacement and an independent rotation vector, which were associated with stresses and couple stresses through constitutive relations. Due to rotation, couple stress theory was able to explain the dispersive nature of the waves, which was not captured by classical theory of elasticity. Hadjesfandiari and Dargush (2011) have proposed a consistent couple stress theory involving one length scale parameter η called couple stress coefficient and two Lame parameters λ, μ. Here the parameter η further depends upon characteristic length parameter (l), which was absent in classical theory of elasticity. It was pointed out by Lakes (1991) that this length scale parameter is of the order of average cell size of the material in cellular solids.

Many researchers have used couple stress theory to study the problems of wave propagation in elastic media under different conditions (Sengupta, Ghosh, 1974a; 1974b; Sengupta, Benerji, 1978; Das *et al.*, 1991; Georgiadis, Velgaki, 2003; Kocaturk, Akbas, 2013). Sharma and Kumar (2014) have studied the role of characteristic length in velocity dispersion in an elastic plate using consistent couple stress theory.

The effects of gravity on wave propagation were firstly studied by Bromwich (1898) by treating gravitational force as a body force. Love (1911) studied the effects of gravity on various problems of wave propagation. He concluded that Rayleigh wave velocity gets significantly affected due to the presence of gravitational field, when the wavelength of waves is large. Plona *et al.* (1975) studied Rayleigh and Lamb waves at liquid-solid boundaries. Bhattacharyya and De (1977) studied surface waves in a viscoelastic media under the effect of gravity. They briefly investigated various surface waves and showed that results are in agreement with classical theory, when effects of gravity and viscosity are neglected. Qi (1994) studied the influence of viscous fluid loading on the propagation of leaky Rayleigh wave in the presence of heat conduction effects. Zhu *et al.* (2004) studied the propagation of leaky Rayleigh and Scholte waves at the fluid-solid interface subjected to transient point loading by using integral transform technique. Georgiadis *et al.* (2004) applied theory of dipolar gradient elasticity to analyze Rayleigh type waves propagating along the surface of a half-space and they showed that these waves are indeed dispersive at high frequencies. Sharma *et al.* (2008) studied the propagation of Rayleigh surface waves in microstretch thermoelastic continua under inviscid fluid loadings in context of classical and non- classical theories of elasticity. Danicki (2010) studied Rayleigh waves in an isotropic body with deep periodic grooves and he showed that surface wave existence and reflection depends on both groove depth and their period. Kumar *et al.* (2014) studied Rayleigh waves in isotropic microstretch thermoelastic diffusion solid half space. They observed the effects of relaxation times on the phase velocity, attenuation coefficient, normal stress, tangential stress, couple stress, microstress, temperature change, and mass concentration. Vinh *et al.* (2014) studied Rayleigh waves in an isotropic elastic half space coated by a thin isotropic elastic layer with smooth contact. By using effective boundary condition method they derived secular equation of fourth-order in terms of dimensionless thickness of the layer. Kakar and Kakar (2014) studied propagation of surface waves in electro-magneto thermoelastic orthotropic granular non-homogeneous medium subjected to gravity and initial compression. They concluded that phase velocity of Stoneley and Rayleigh waves not only depends on gravity field but also on non-homogeneity, magnetic field, electric field, temperature, initial stress and granular notations of the material. Vavva *et al.* (2014) have applied Mindlin's Form II gradient elasticity to determine Rayleigh wave dispersion in bone mimicking half-space. Tanuma *et al.* (2015) studied dispersion of Rayleigh waves in weakly anisotropic media with vertically-inhomogeneous initial stress. They derived high-frequency asymptotic formula for dispersion relations of Rayleigh waves that propagate in various directions along the free surface of a vertically-inhomogeneous, prestressed and anisotropic half-space. Kaur *et al.* (2016) studied effects of reinforcement, gravity and liquid loadings on propagation of Rayleigh type waves. They derived secular equation in closed form for the propagation of Rayleigh waves and made comparisons of the effects of various parameters involved in the problem on the propagation of Rayleigh waves. Recently, Sharma and Kumar (2017) have applied consistent couple stress theory and studied the effects of liquid loading on the propagation of leaky Rayleigh waves.

The purpose of this article is to study the effect of gravity on the propagation of Rayleigh waves in a couple stress solid half-space loaded with liquid layer. Elastic constants of cortical bone which exhibits microstructures are used for numerical calculations. Bone is a composite material exhibiting different microstructures at different length scales. Consistent couple stress theory (Hadjesfandiari, Dargush, 2011) and the model of gravity given in Bhattacharyya and De (1977) is used to study the impact of characteristic length parameter, gravity and liquid loadings on velocity dispersion of leaky Rayleigh waves propagating in a couple stress half space under the effects of gravity and loaded with homogeneous inviscid liquid layer of finite thickness (H) or a liquid half space.

2. Formulation and solution of the problem

Consider a layer of inviscid liquid of finite thickness (H) or a liquid half space lying over a couple stress half space under the effects of gravity (g). It is assumed that there is no reflection from the inner layers of the liquid medium. The origin of the coordinate system $O(x, y, z)$ lies on the interfacial surface joining solid half space and liquid medium. It is assumed that z-axis of the coordinate system is pointing vertically downwards into solid half space and wave is assumed to propagate in the direction of x-axis. All the particles along a line parallel to y-axis are assumed to be equally displaced, thus there is no variation of the fields in either media in the direction of y-axis.

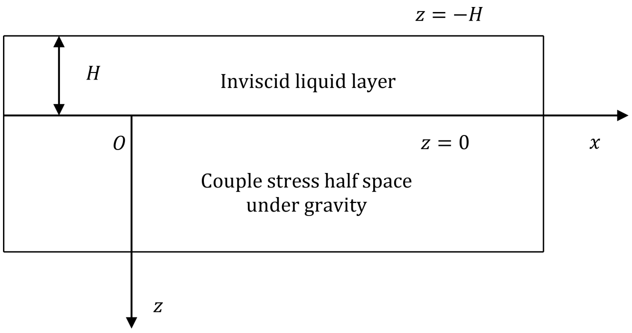

Fig. 1. Geometry of the problem.

2.1. *Couple stress substrate under gravity*

The basic governing equations of motion of couple stress theory (Hadjesfandiari, Dargush, 2011) following Bhattacharyya and De (1977) for the effects of gravity, are given by

$$\frac{(\lambda+2\mu)}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right) -\frac{\mu}{\rho}\left(1-l^2\nabla^2\right)\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 w}{\partial z\partial x}\right) +g\frac{\partial w}{\partial x}=\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

$$\frac{(\lambda+2\mu)}{\rho}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 w}{\partial y\partial z}\right) -\frac{\mu}{\rho}\left(1-l^2\nabla^2\right)\left(\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 w}{\partial z\partial y}\right) +g\frac{\partial w}{\partial y}=\frac{\partial^2 v}{\partial t^2}, \tag{2}$$

$$\frac{(\lambda+2\mu)}{\rho}\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}+\frac{\partial^2 w}{\partial z^2}\right) -\frac{\mu}{\rho}\left(1-l^2\nabla^2\right)\left(\frac{\partial^2 u}{\partial x\partial z}-\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 v}{\partial y\partial z}\right) -g\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

where (u,v,w) are the displacement components, g is gravity, l is characteristic length parameter, ρ is the density, λ and μ are Lame parameters. We confine our discussion to two dimensional medium so, we take $\mathbf{u}=(u,0,w)$ and $\frac{\partial}{\partial y}\equiv 0$, in this case Eq. (2) will be identically satisfied.

The constitutive relations for couple stress half space are given by

$$\sigma_{ji} = \lambda u_{k,k}\delta_{ij}+\mu\left(u_{i,j}+u_{j,i}\right)-\eta\nabla^2\left(u_{i,j}-u_{j,i}\right), \tag{4}$$

$$\mu_{ji} = 4\eta\left(\omega_{i,j}-\omega_{j,i}\right), \quad \text{where} \quad \omega_i=\frac{1}{2}\epsilon_{ijk}u_{k,j}. \tag{5}$$

Here, σ_{ji} is the non-symmetric force-stress tensor, μ_{ji} is skew symmetric couple-stress tensor, δ_{ij} is Kronecker's delta, $\eta=\mu l^2$ is couple-stress coefficient, ϵ_{ijk} is permutation tensor.

Now we introduce potential functions ϕ and $\boldsymbol{\psi}=(0,\psi,0)$ in the solid such that $\mathbf{u}=\nabla\phi+\nabla\times\boldsymbol{\psi}$, so we get

$$u=\frac{\partial\phi}{\partial x}-\frac{\partial\psi}{\partial z} \qquad \text{and} \qquad w=\frac{\partial\phi}{\partial z}+\frac{\partial\psi}{\partial x}, \tag{6}$$

where ϕ and ψ are potential function of longitudinal and shear waves in the solid half space. Using these values of potential functions in Eqs. (1) and (3), we get

$$\frac{\partial}{\partial x}\left[C_1^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right)-\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\psi}{\partial x}\right] -\frac{\partial}{\partial z}\left[C_2^2\left(1-l^2\nabla^2\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right) -\frac{\partial^2\psi}{\partial t^2}-g\frac{\partial\phi}{\partial x}\right]=0, \tag{7}$$

$$\frac{\partial}{\partial z}\left[C_1^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right)-\frac{\partial^2\phi}{\partial t^2}+g\frac{\partial\psi}{\partial x}\right] +\frac{\partial}{\partial x}\left[C_2^2\left(1-l^2\nabla^2\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right) -\frac{\partial^2\psi}{\partial t^2}-g\frac{\partial\phi}{\partial x}\right]=0, \tag{8}$$

where

$$C_1^2=\frac{(\lambda+2\mu)}{\rho}, \qquad C_2^2=\frac{\mu}{\rho}.$$

From Eqs. (7) and (8), we get following two partial differential equations

$$C_1^2\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}\right)+g\frac{\partial\psi}{\partial x} = \frac{\partial^2\phi}{\partial t^2}, \tag{9}$$

$$C_2^2\left(1-l^2\nabla^2\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}\right)-g\frac{\partial\phi}{\partial x} = \frac{\partial^2\psi}{\partial t^2}. \tag{10}$$

Now, we consider solutions as

$$\phi = f(z)\mathrm{e}^{i\xi(x-ct)}, \tag{11}$$

$$\psi = h(z)\mathrm{e}^{i\xi(x-ct)}. \tag{12}$$

Using above mentioned solutions, Eqs. (9) and (10) reduce to following two differential equations

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2}-\left(\xi^2-\frac{\xi^2c^2}{C_1^2}\right)f(z)+\frac{gi\xi}{C_1^2}h(z) = 0, \tag{13}$$

$$\frac{\mathrm{d}^4 h(z)}{\mathrm{d}z^4}-\left(2\xi^2+\frac{1}{l^2}\right)\frac{\mathrm{d}^2 h(z)}{\mathrm{d}z^2} +\left(\xi^4+\frac{\xi^2}{l^2}\left(1-\frac{c^2}{C_2^2}\right)\right)h(z)+\frac{gi\xi}{l^2C_2^2}f(z) = 0. \tag{14}$$

By solving Eqs. (13) and (14), we get following differential equation

$$\frac{\mathrm{d}^6 f(z)}{\mathrm{d}z^6} - (P_1+P_2)\frac{\mathrm{d}^4 f(z)}{\mathrm{d}z^4} + (P_1P_2+P_3)\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} - (P_1P_3 - P_4) f(z) = 0, \quad (15)$$

where

$$P_1 = \xi^2\left(1-\frac{c^2}{C_1^2}\right), \qquad P_2 = \left(2\xi^2+\frac{1}{l^2}\right),$$

$$P_3 = \xi^4 + \frac{\xi^2}{l^2} - \frac{\xi^2 c^2}{l^2 C_2^2}, \qquad P_4 = \frac{g^2\xi^2}{l^2 C_1^2 C_2^2}.$$

By solving Eq. (15), we get

$$f(z) = m_1 e^{-t_{11}z} + m_2 e^{-t_{22}z} + m_3 e^{-t_{33}z}, \quad (16)$$

$$\phi = \left(m_1 e^{-t_{11}z} + m_2 e^{-t_{22}z} + m_3 e^{-t_{33}z}\right)\mathrm{e}^{i\xi(x-ct)}, \quad (17)$$

where m_1, m_2 and m_3 are unknown constants and

$$t_{11} = \sqrt{2a-a_1}, \qquad t_{22} = \sqrt{-a-b\sqrt{3}-a_1},$$

$$t_{33} = \sqrt{-a+b\sqrt{3}-a_1}, \qquad 2a-a_1>0,$$

$$-a-b\sqrt{3}-a_1>0, \qquad -a+b\sqrt{3}-a_1>0,$$

$$a = r^{\frac{1}{3}}\cos\left(\frac{\theta}{3}\right), \qquad b = r^{\frac{1}{3}}\sin\left(\frac{\theta}{3}\right),$$

$$r = \sqrt{-H_1^3}, \qquad \tan\theta = \frac{-\sqrt{-G_1^2-4H_1^3}}{G_1},$$

$$G_1 = a_3 - 3a_1a_2 + 2a_1^3, \qquad H_1 = a_2 - a_1^2,$$

$$G_1^2 + 4H_1^3 < 0, \qquad a_1 = \frac{-(P_1+P_2)}{3},$$

$$a_2 = \frac{P_1P_2+P_3}{3}, \qquad a_3 = P_4 - P_1P_3.$$

Using Eq. (16) in Eq. (13), we get

$$h(z) = \alpha_1 m_1\left[-t_{11}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right]\mathrm{e}^{-t_{11}z} + \alpha_1 m_2\left[-t_{22}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right]\mathrm{e}^{-t_{22}z} + \alpha_1 m_3\left[-t_{33}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right]\mathrm{e}^{-t_{33}z}, \quad (18)$$

$$\psi = \left[\alpha_1 m_1\left[-t_{11}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right]\mathrm{e}^{-t_{11}z} + \alpha_1 m_2\left[-t_{22}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right]\mathrm{e}^{-t_{22}z} + \alpha_1 m_3\left[-t_{33}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right]\mathrm{e}^{-t_{33}z}\right] \cdot \mathrm{e}^{i\xi(x-ct)}, \quad (19)$$

$$\psi = \left[m_1\beta_1 e^{-t_{11}z} + m_2\beta_2 e^{-t_{22}z} + m_3\beta_3 e^{-t_{33}z}\right]\mathrm{e}^{i\xi(x-ct)}, \quad (20)$$

where

$$\alpha_1 = \frac{C_1^2}{gi\xi},$$

$$\beta_1 = \alpha_1\left[-t_{11}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right] = \frac{C_1^2}{gi\xi}\left[-t_{11}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right],$$

$$\beta_2 = \alpha_1\left[-t_{22}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right] = \frac{C_1^2}{gi\xi}\left[-t_{22}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right],$$

$$\beta_3 = \alpha_1\left[-t_{33}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right] = \frac{C_1^2}{gi\xi}\left[-t_{33}^2 + \left(1-\frac{c^2}{C_1^2}\right)\xi^2\right].$$

2.2. Inviscid liquid layer

Equation of motion for inviscid liquid layer is

$$\nabla(\nabla\cdot\mathbf{u}_L) = \frac{1}{C_L^2}\frac{\partial^2\mathbf{u}_L}{\partial t^2}, \quad (21)$$

where $\mathbf{u}_L = (u_L, v_L, w_L)$ is displacement components in liquid layer, $C_L^2 = \frac{\lambda_L}{\rho_L}$, C_L is the velocity of sound in the liquid, λ_L is the bulk modulus, ρ_L is the density of the liquid.

In the liquid medium, we have

$$u_L = \frac{\partial\phi_L}{\partial x} - \frac{\partial\psi_L}{\partial z} \quad \text{and} \quad w_L = \frac{\partial\phi_L}{\partial z} + \frac{\partial\psi_L}{\partial x}. \quad (22)$$

Here ϕ_L and ψ_L are the scalar and vector potentials. Inviscid liquid does not support shear motion, so shear modulus of liquid vanishes that is $\mu_L = 0$ or $\psi_L = 0$.

Using this condition in Eq. (21), equation of motion for inviscid liquid layer becomes

$$\nabla^2\phi_L = \frac{1}{C_L^2}\frac{\partial^2\phi_L}{\partial t^2} \quad (23)$$

and stresses σ_{ij}^L in case of liquid medium are given by

$$\sigma_{ij}^L = \lambda_L\left(\frac{\partial u_L}{\partial x} + \frac{\partial w_L}{\partial z}\right)\delta_{ij}. \quad (24)$$

We take the solution of Eq. (23) as

$$\phi_L = k(z)\mathrm{e}^{i\xi(x-ct)}. \quad (25)$$

By putting above solution in Eqs. (23), we get

$$\phi_L = \left(m_4\mathrm{e}^{t_{44}z} + m_5\mathrm{e}^{-t_{44}z}\right)\mathrm{e}^{i\xi(x-ct)}, \quad (26)$$

where

$$t_{44}^2 = \xi^2 \left(1 - \frac{c^2}{C_L^2}\right).$$

For propagation of Rayleigh waves in the half space $z \geq 0$ underlying a liquid layer of finite thickness (H) or a liquid half-space, we choose the solution as

$$\phi_L = \begin{cases} m_4 \sinh\left\{t_{44}\left(z+H\right)\right\} \mathrm{e}^{i\xi(x-ct)}, \\ \qquad \text{for the liquid layer, } -H \leq z \leq 0, \\ m_4 \mathrm{e}^{t_{44} z} \mathrm{e}^{i\xi(x-ct)}, \\ \qquad \text{for the liquid half-space.} \end{cases} \tag{27}$$

3. Boundary conditions

The boundary conditions to be satisfied at the solid-liquid interface are:

(i) $\sigma_{zz} = \sigma_{zz}^L$, at the interfacial surface $z = 0$ which leads to

$$\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} = \lambda_L\left(\frac{\partial u_L}{\partial x} + \frac{\partial w_L}{\partial z}\right)$$

at $z = 0$,

(ii) $\sigma_{zx} = 0$, at $z = 0$ that is

$$\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) - \mu l^2\left(\frac{\partial^3 u}{\partial x^2 \partial z} - \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 u}{\partial z^3} - \frac{\partial^3 w}{\partial z^2 \partial x}\right) = 0$$

at $z = 0$,

(iii) $w = w_L$, at $z = 0$,

(iv) $\mu_{zy} = 0$, at $z = 0$, that is

$$2\eta\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial z \partial x}\right) = 0$$

at $z = 0$.

4. Derivation of secular equation

4.1. Rayleigh waves in a couple stress half space loaded with inviscid liquid half space under gravity

Using above mentioned boundary conditions together with Eqs. (17), (20) and (27), we get following four equations

$$\mu\left\{\frac{C_1^2}{C_2^2}\left(t_{11}^2 - \xi^2\right) + 2\xi^2 - 2\beta_1 t_{11} i\xi\right\} m_1 + \mu\left\{\frac{C_1^2}{C_2^2}\left(t_{22}^2 - \xi^2\right) + 2\xi^2 - 2\beta_2 t_{22} i\xi\right\} m_2 + \mu\left\{\frac{C_1^2}{C_2^2}\left(t_{33}^2 - \xi^2\right) + 2\xi^2 - 2\beta_3 t_{33} i\xi\right\} m_3 - \lambda_L\left(t_{44}^2 - \xi^2\right) m_4 = 0, \tag{28}$$

$$\left(\beta_1 i\xi - t_{11}\right) m_1 + \left(\beta_2 i\xi - t_{22}\right) m_2 + \left(\beta_3 i\xi - t_{33}\right) m_3 - t_{44} m_4 = 0, \tag{29}$$

$$\left(\beta_1 t_{11}\xi^2 - \beta_1 t_{11}^3\right) m_1 + \left(\beta_2 t_{22}\xi^2 - \beta_2 t_{22}^3\right) m_2 + \left(\beta_3 t_{33}\xi^2 - \beta_3 t_{33}^3\right) m_3 = 0, \tag{30}$$

$$\left\{l^2\left(\xi^4 + t_{11}^4 - 2t_{11}^2\xi^2\right)\beta_1 - 2t_{11} i\xi - \beta_1\xi^2 - \beta_1 t_{11}^2\right\} m_1 + \left\{l^2\left(\xi^4 + t_{22}^4 - 2t_{22}^2\xi^2\right)\beta_2 - 2t_{22} i\xi - \beta_2\xi^2 - \beta_2 t_{22}^2\right\} m_2 + \left\{l^2\left(\xi^4 + t_{33}^4 - 2t_{33}^2\xi^2\right)\beta_3 - 2t_{33} i\xi - \beta_3\xi^2 - \beta_3 t_{33}^2\right\} m_3 = 0. \tag{31}$$

For non-trivial solution of the above four homogeneous linear equations, determinant of four unknown coefficients m_1, m_2, m_3, m_4 must be zero. By imposing this condition we get following determinant

$$\begin{vmatrix} E_1 & E_2 & E_3 & E_4 \\ \beta_1 i\xi - t_{11} & \beta_2 i\xi - t_{22} & \beta_3 i\xi - t_{33} & -t_{44} \\ G_1 & G_2 & G_3 & 0 \\ F_1 & F_2 & F_3 & 0 \end{vmatrix} = 0, \tag{32}$$

where

$$E_1 = \mu\left\{\frac{C_1^2}{C_2^2}\left(t_{11}^2 - \xi^2\right) + 2\xi^2 - 2\beta_1 t_{11} i\xi\right\},$$

$$E_2 = \mu\left\{\frac{C_1^2}{C_2^2}\left(t_{22}^2 - \xi^2\right) + 2\xi^2 - 2\beta_2 t_{22} i\xi\right\},$$

$$E_3 = \mu\left\{\frac{C_1^2}{C_2^2}\left(t_{33}^2 - \xi^2\right) + 2\xi^2 - 2\beta_3 t_{33} i\xi\right\},$$

$$E_4 = -\lambda_L(t_{44}^2 - \xi^2),$$

$$G_1 = \beta_1 t_{11}\xi^2 - \beta_1 t_{11}^3,$$

$$G_2 = \beta_2 t_{22}\xi^2 - \beta_2 t_{22}^3,$$

$$G_3 = \beta_3 t_{33}\xi^2 - \beta_3 t_{33}^3,$$

$$F_1 = \left\{l^2\left(\xi^4 + t_{11}^4 - 2t_{11}^2\xi^2\right)\beta_1 - 2t_{11} i\xi - \beta_1\xi^2 - \beta_1 t_{11}^2\right\},$$

$$F_2 = \left\{l^2\left(\xi^4 + t_{22}^4 - 2t_{22}^2\xi^2\right)\beta_2 - 2t_{22} i\xi - \beta_2\xi^2 - \beta_2 t_{22}^2\right\},$$

$$F_3 = \left\{l^2\left(\xi^4 + t_{33}^4 - 2t_{33}^2\xi^2\right)\beta_3 - 2t_{33} i\xi - \beta_3\xi^2 - \beta_3 t_{33}^2\right\}.$$

By solving the determinant we get following dispersion equation for Rayleigh waves in a couple stress substrate loaded with liquid half space under the effect of gravity

$$\begin{aligned}
&\lambda_L\left(t_{44}^2-\xi^2\right)\{(-t_{11}+\beta_{11}\xi)(U_1+V_{11})\\
&+(t_{22}-\beta_{22}\xi)(U_2+V_{22})+(-t_{33}+\xi\beta_{33})(U_3+V_{33})\}\\
&-t_{44}\mu\left\{\left(\frac{C_1^2}{C_2^2}\left(t_{11}^2-\xi^2\right)+2\xi^2-2\beta_{11}t_{11}\xi\right)(U_1+V_{11})\right.\\
&-\left(\frac{C_1^2}{C_2^2}\left(t_{22}^2-\xi^2\right)+2\xi^2-2\beta_{22}t_{22}\xi\right)(U_2+V_{22})\\
&\left.+\left(\frac{C_1^2}{C_2^2}\left(t_{33}^2-\xi^2\right)+2\xi^2-2\beta_{33}t_{33}\xi\right)(U_3+V_{33})\right\}=0, \qquad (33)
\end{aligned}$$

where

$$\beta_1=\frac{\beta_{11}}{i},\qquad \beta_2=\frac{\beta_{22}}{i},\qquad \beta_3=\frac{\beta_{33}}{i}.$$

4.2. Rayleigh waves in a couple stress half space loaded with inviscid liquid layer of finite thickness under gravity

Using above mentioned boundary conditions together with Eqs. (17), (20) and (27), we get following four equations

$$\begin{aligned}
&\mu\left\{\frac{C_1^2}{C_2^2}\left(t_{11}^2-\xi^2\right)+2\xi^2-2\beta_1t_{11}i\xi\right\}m_1\\
&+\mu\left\{\frac{C_1^2}{C_2^2}\left(t_{22}^2-\xi^2\right)+2\xi^2-2\beta_2t_{22}i\xi\right\}m_2\\
&+\mu\left\{\frac{C_1^2}{C_2^2}\left(t_{33}^2-\xi^2\right)+2\xi^2-2\beta_3t_{33}i\xi\right\}m_3\\
&-\lambda_L\left(t_{44}^2-\xi^2\right)\sinh\left(t_{44}H\right)m_4=0, \qquad (34)
\end{aligned}$$

$$\begin{aligned}
&(\beta_1i\xi-t_{11})\,m_1+(\beta_2i\xi-t_{22})\,m_2\\
&+(\beta_3i\xi-t_{33})\,m_3-t_{44}\cosh\left(t_{44}H\right)m_4=0, \qquad (35)
\end{aligned}$$

$$\begin{aligned}
&\left(\beta_1t_{11}\xi^2-\beta_1t_{11}^3\right)m_1\\
&+\left(\beta_2t_{22}\xi^2-\beta_2t_{22}^3\right)m_2\\
&+\left(\beta_3t_{33}\xi^2-\beta_3t_{33}^3\right)m_3=0, \qquad (36)
\end{aligned}$$

$$\begin{aligned}
&\left\{l^2\left(\xi^4+t_{11}^4-2t_{11}^2\xi^2\right)\beta_1-2t_{11}i\xi\right.\\
&\left.-\beta_1\xi^2-\beta_1t_{11}^2\right\}m_1\\
&+\left\{l^2\left(\xi^4+t_{22}^4-2t_{22}^2\xi^2\right)\beta_2\right.\\
&\left.-2t_{22}i\xi-\beta_2\xi^2-\beta_2t_{22}^2\right\}m_2\\
&+\left\{l^2\left(\xi^4+t_{33}^4-2t_{33}^2\xi^2\right)\beta_3-2t_{33}i\xi\right.\\
&\left.-\beta_3\xi^2-\beta_3t_{33}^2\right\}m_3=0. \qquad (37)
\end{aligned}$$

For non-trivial solution of the above four homogeneous linear equations, determinant of four unknown coefficients m_1, m_2, m_3, m_4 must be zero. By imposing this condition we get following determinant

$$\begin{vmatrix} E_1 & E_2 & E_3 & E_{44}\\ H_1 & H_2 & H_3 & H_4\\ G_1 & G_2 & G_3 & 0\\ F_1 & F_2 & F_3 & 0\end{vmatrix}=0, \qquad (38)$$

where

$$\begin{aligned}
E_{44}&=-\lambda_L\left(t_{44}^2-\xi^2\right)\sinh\left(t_{44}H\right),\\
H_1&=\beta_1i\xi-t_{11},\\
H_2&=\beta_2i\xi-t_{22},\\
H_3&=\beta_3i\xi-t_{33},\\
H_4&=-t_{44}\cosh\left(t_{44}H\right),\\
G_1&=\beta_1t_{11}\xi^2-\beta_1t_{11}^3,\\
G_2&=\beta_2t_{22}\xi^2-\beta_2t_{22}^3,\\
G_3&=\beta_3t_{33}\xi^2-\beta_3t_{33}^3.
\end{aligned}$$

By solving the determinant, we get following dispersion equation for Rayleigh waves in a couple stress substrate loaded with inviscid liquid layer of finite thickness under the effect of gravity

$$\begin{aligned}
&\lambda_L\left(t_{44}^2-\xi^2\right)\tanh(t_{44}H)\left\{(-t_{11}+\beta_{11}\xi)(U_1+V_{11})\right.\\
&\left.+(t_{22}-\beta_{22}\xi)(U_2+V_{22})+(-t_{33}+\xi\beta_{33})(U_3+V_{33})\right\}\\
&-t_{44}\mu\left\{\left(\frac{C_1^2}{C_2^2}\left(t_{11}^2-\xi^2\right)+2\xi^2-2\beta_{11}t_{11}\xi\right)(U_1+V_{11})\right.\\
&-\left(\frac{C_1^2}{C_2^2}\left(t_{22}^2-\xi^2\right)+2\xi^2-2\beta_{22}t_{22}\xi\right)(U_2+V_{22})\\
&\left.+\left(\frac{C_1^2}{C_2^2}\left(t_{33}^2-\xi^2\right)+2\xi^2-2\beta_{33}t_{33}\xi\right)(U_3+V_{33})\right\}=0. \qquad (39)
\end{aligned}$$

In Eq. (39), if $H\to\infty$, that is liquid layer changes to liquid half space, then $\tanh(t_{44}H)\to 1$, and Eq. (39) reduces to Eq. (33) for leaky Rayleigh waves in a couple stress half space under the loading of liquid half space with effects of gravity.

5. Numerical results and discussion

Here to investigate the effects of inner microstructure of half space in terms of characteristic length on the propagation of leaky Rayleigh waves the material of solid elastic half space is assumed to have properties similar to cortical bone. Following Vavva *et al.* (2009), the material properties of couple stress substrate are

E = Young's modulus = $\dfrac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$	14 GPa
ν = Poisson ratio = $\dfrac{\lambda}{2(\lambda+\mu)}$	0.37
Density = ρ	1500 kg/m^3

The values of bulk longitudinal and shear velocities are $C_1 = 4063$ m/s, $C_2 = 1846$ m/s, respectively. Microstructure of bone ranges from 10 to 500 μm (Vavva *et al.*, 2009), so five different values of characteristic length (l), lying in the above mentioned range viz., $l = 0.00009$ m, 0.0001 m, 0.0002 m, 0.0003 m, 0.0005 m are considered. The liquid medium used is inviscid liquid with $C_L = 1.5 \cdot 10^3$ m/s and density $\rho_L = 1000$ kg/m^3.

5.1. Effects of gravity

Figure 2 shows the general trend for dispersion curve of Rayleigh waves in the considered model. Here, non-dimensional phase velocity (c/C_2) of Rayleigh waves is plotted against non- dimensional wave number (ξH) by taking characteristic length parameter $l = 0.0001$ m under the combined effects of gravity ($g = 9.8$ m/s^2) and liquid loadings with thickness of liquid layer as $H = 0.002$ m. Rayleigh waves are observed to be dispersive in the considered model. It is observed that Rayleigh wave velocity is higher for small wave number range and then it decreases down with the increasing values of the wave number and for the higher values of wave number it becomes almost constant.

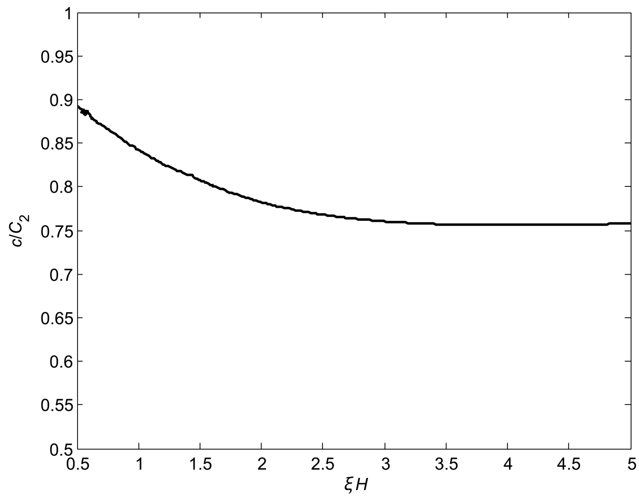

Fig. 2. Phase velocity profile of Rayleigh waves in a couple stress substrate with wave number under the effect of gravity and liquid loadings.

Figures 3a and 3b show the effects of gravity on Rayleigh wave velocity in a couple stress half space loaded with liquid layer of finite thickness. In Fig. 3a wave number (ξ) is normalized using thickness (H) of the layer, whereas for Fig. 3b characteristic length (l) is used for the same. Here, the value of characteristic length parameter is $l = 0.0002$ m and the thickness of liquid layer is $H = 0.002$ m. Comparison is made by taking gravity parameter $g = 9.8$ m/s^2 for G1 curve and $g = 0.0$ m/s^2 (without gravity) for G2 curve. It is observed that Rayleigh waves propagate with lower

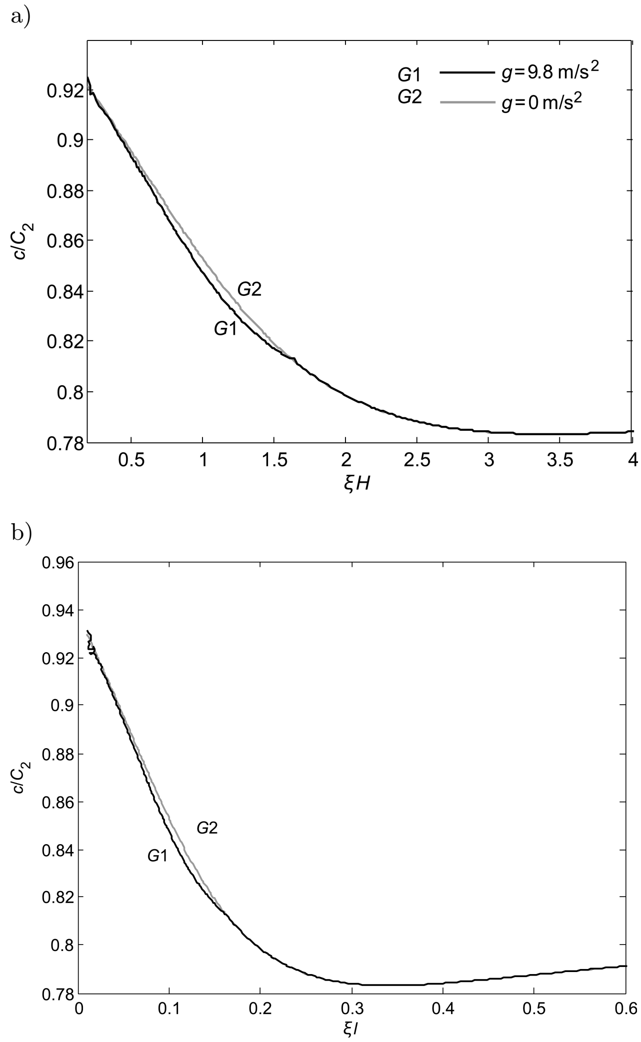

Fig. 3. Phase velocity profile of Rayleigh waves against wave number showing effects of gravity: a) normalised using H, b) normalised using l.

phase velocity under the effects of gravity. This trend is observed only for lower values of wave number, no such difference is observed for the higher values of wave number. This finding is quite similar to the one observed by Love (1911).

5.2. Effects of thickness of liquid layer

Effects of thickness of liquid layer on phase velocity of Rayleigh waves are shown in Fig. 4a and 4b. For the curves in Fig. 4a and 4b the value of gravity is $g = 9.8$ m/s^2, characteristic length parameter is $l = 0.0001$ m and the value of thickness of liquid layer is $H = 0.001$ m, 0.002 m, 0.004 m for $H1$, $H2$ and $H3$ curves respectively. From the figures it is clear that the thickness of liquid layer has a prominent effect on phase velocity of Rayleigh waves and it is seen that increasing value of thickness of liquid layer has an adverse effect on phase velocity of Rayleigh waves. Phase velocity

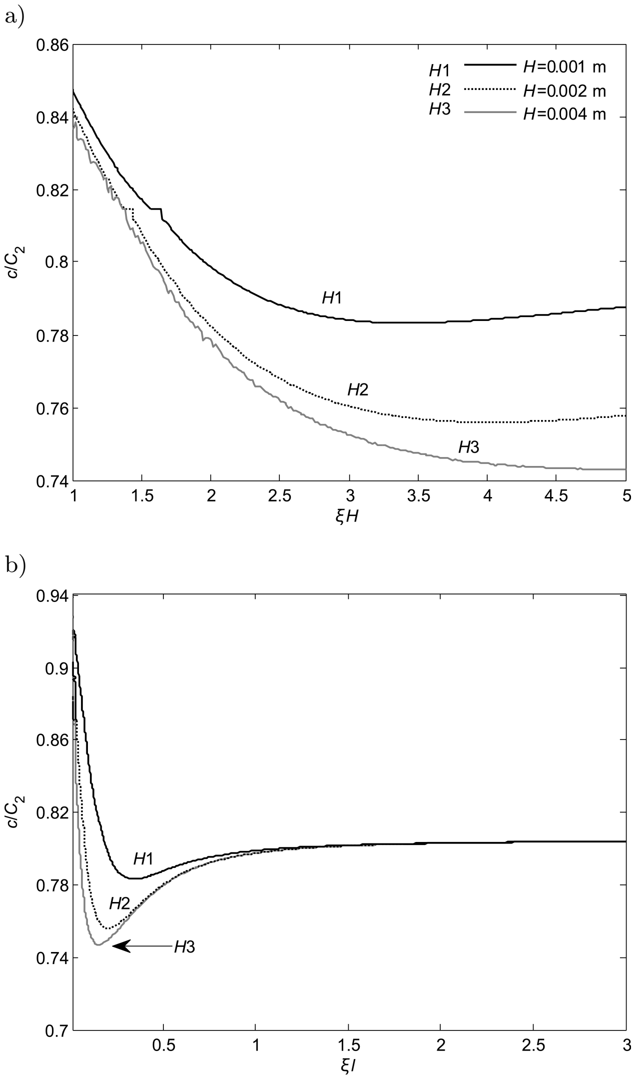

Fig. 4. Phase velocity profile of Rayleigh waves under gravity against wave number showing effects of thickness of liquid layer: a) normalised using H, b) normalised using l.

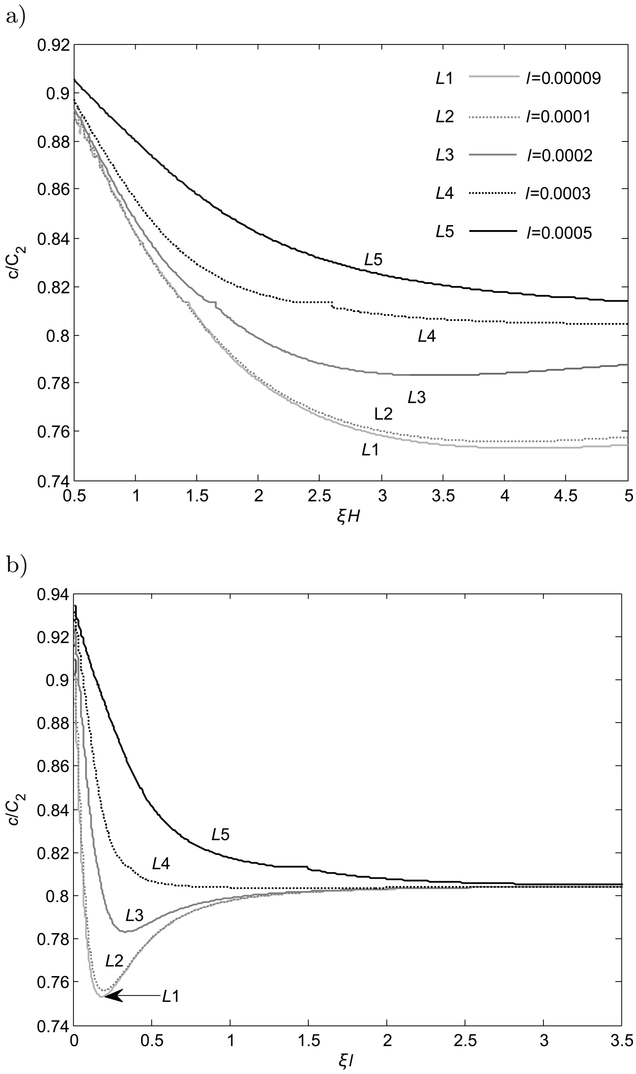

Fig. 5. Phase velocity profile of Rayleigh waves under gravity against wave number showing effects of characteristic length: a) normalised using H, b) normalised using l.

is found to be decreasing with the increasing value of thickness of liquid layer. Again, thickness (H) of the layer is used for normalizing the wave number (ξ) in Fig. 4a and characteristic length (l) is used for the same in Fig. 4b.

5.3. Effects of characteristic length of couple stress half space

Figures 5a and 5b show the effects of microstructural parameter characteristic length (l) of the underlying couple stress substrate on phase velocity of Rayleigh waves in the presence of both gravity and liquid loadings. The values of the various parameters are $H = 0.002$ m, $g = 9.8$ m/s^2 and five values of characteristic length parameter are $l = 0.00009$ m, 0.0001 m, 0.0002 m, 0.0003 m, 0.0005 m for the curves $L1$, $L2$, $L3$, $L4$, $L5$ respectively. From the figures it can be concluded that increasing value of characteristic length favours phase velocity of Rayleigh waves. For a same wave number the value of phase velocity is higher with the increasing value of characteristic length parameter (l).

6. Conclusion

The behaviour of Rayleigh waves propagating in a couple stress half space under the effects of gravity and loaded with homogeneous inviscid liquid layer of finite thickness or a liquid half space has been studied in this paper. The properties of Rayleigh waves get affected by thickness of loaded liquid and properties of underlying solid half space. Following observations can be made from the present analysis:

1) Rayleigh waves are found to be dispersive in nature, under the model considered in the problem.

It can be observed that for the small values of wave number the phase velocity of Rayleigh waves is high, which decreases with the increase in wave number and then becomes almost constant for the higher values of wave numbers.

2) It is found that gravity has minor effect on the propagation of Rayleigh wave velocity and these are visible only for the lower values wave number. Phase velocity of Rayleigh waves decreases due to the presence of gravity.
3) The effect of liquid loading is observed by taking three different values of thickness of liquid layer and it is found that thickness of liquid layer has notable effects on phase velocity of Rayleigh waves. It is observed that phase velocity of Rayleigh waves decreases with the increase in the thickness of liquid layer.
4) Effects of characteristic length parameter involved in couple stress theory which measures internal microstructures of the material are also studied on the propagation of Rayleigh waves and it is found that characteristic length parameter has profound effects on Rayleigh wave velocity. Study has been made by considering five different values of characteristic length parameter and it is observed that for a same wave number the value of phase velocity is higher with the increasing value of characteristic length parameter.

Dispersion characteristics of Rayleigh waves obtained in the article may be used to estimate the frequency of excited waves employed to scan the bodies of bedridden patients or other bones related problems. In addition, Rayleigh waves have large number of applications in various engineering and scientific fields like seismology, geophysics and non-destructive testing of structures. A structure surrounded by liquid is a natural occurring phenomenon in the fields like oil exploration at oceanic bed or under water NDT applications.

As the proposed model incorporates microstructural and gravitational effects so, it may provide some modifications to existing methods of the non-destructive techniques which are based on classical theory. This model may also find some possible applications in the fields of biomedical sciences, seismology and geophysics.

References

1. Bhattacharyya S., De S.N. (1977), *Surface waves in viscoelastic media under the influence of gravity*, Australian Journal of Physics, **30**, 347–353.
2. Bromwich T.J.I'A. (1898), *On the influence of gravity on elastic waves, and, in particular, on the vibrations of elastic globe*, Proceedings of the London Mathematical Society, **30**, 98–120.
3. Cosserat E., Cosserat F. (1909), *Theory of deformable bodies* [in French: *Théorie des corps déformables*], A. Hermann et Fils, Paris.
4. Danicki E. (2010), *Rayleigh Waves in an isotropic Body with Deep Periodic Grooves*, Archives of Acoustics, **35**, 1, 67–74.
5. Das T.K., Sengupta P.R., Debnath L. (1991), *Thermo-visco-elastic Rayleigh waves under the influence of couple stress and gravity*, International Journal of Mathematics and Mathematical Sciences, 14, 3, 553–560.
6. Eringen A.C. (1968), *Theory of micropolar elasticity*, [in:] Liebowitz H. (Ed.), *Fracture*, vol. 2, Academic Press, New York, pp. 662–729.
7. Georgiadis H.G., Vardoulakis I., Velgaki E.G. (2004), *Dispersive Rayleigh-wave propagation in microstructured solids characterized by dipolar gradient elasticity*, Journal of Elasticity, **74**, 17–45.
8. Georgiadis H.G., Velgaki E.G. (2003), *High-frequency Rayleigh waves in materials with microstructure and couple-stress effects*, International Journal of Solids and Structures, **40**, 2501–2520.
9. Hadjesfandiari A.R., Dargush G.F. (2011), *Couple stress theory for solids*, International Journal of Solids and Structures, **48**, 2496–2510.
10. Kakar R., Kakar S. (2014), *Electro-magneto-thermoelastic surface waves in non-homogeneous orthotropic granular half space*, Geomechanics and Engineering, **7**, 1, 1–36.
11. Kaur T., Sharma S.K., Singh A.K. (2016), *Effect of reinforcement, gravity and liquid loading on Rayleigh-type wave propagation*, Meccanica, **51**, 10, 2449–2458.
12. Kocaturk T., Akbas S.D. (2013), *Wave propagation in a microbeam based on the modified couple stress theory*, Structural Engineering and mechanics, **46**, 3, 417–431.
13. Koiter W.T. (1964), *Couple stresses in the theory of elasticity, I and II*, Koninklijke Nederlandse Akademie van Wetenschappen, Series B, **67**, 17–44.
14. Kumar R., Ahuja S., Garg S.K. (2014), *Rayleigh waves in isotropic microstretch thermoelastic diffusion solid half space*, Latin American Journal of Solids and Structures, **11**, 299–319.
15. Lakes R.S. (1991), *Experimental micro mechanics methods for conventional and negative poisson's ratio cellular solids as cosserat continua*, Journal of Engineering Materials and Technology, **113**, 148–155.
16. Love A.E.H. (1911), *Some problems of geodynamics*, Cambridge University Press, London.
17. Mindlin R.D., Tiersten H.F. (1962), *Effects of couple-stresses in linear elasticity*, Archive for Rational Mechanics and Analysis, **11**, 415–448.
18. Nowacki W. (1974), *Micropolar Elasticity, International Center for Mechanical Sciences, Courses and Lectures No. 151*, Udine, Springer-Verlag, Wien–New York.

19. Plona T.J., Behravesh M., Mayer W.G. (1975), *Rayleigh and Lamb waves at liquid-solid boundaries*, Ultrasonics,**13**, 171–174.

20. Qi Q. (1994), *Attenuated leaky Lamb waves*, Journal of Acoustical Society of America, **95**, 3222–3231.

21. Sengupta P.R., Benerji D.K. (1978), *Effects of couple-stresses on propagation of waves in an elastic layer immersed in an infinite liquid*, International Journal of Pure and Applied Mathematics, **9**, 17–28.

22. Sengupta P.R., Ghosh B. (1974a), *Effects of couple stresses on the surface waves in elastic media*, Gerlands Beitrage zur Geophysik, Leipzig, **83**, 309–318.

23. Sengupta P.R., Ghosh B. (1974b), *Effects of couple stresses on the propagation of waves in an elastic layer*, Pure and Applied Geophysics, **112**, 331–338.

24. Sharma J.N., Kumar S., Sharma Y.D. (2008), *Propagation of Rayleigh surface waves in microstretch thermoelastic continua under inviscid fluid loadings*, Journal of Thermal Stresses, **31**, 18–39.

25. Sharma V., Kumar S. (2014), *Velocity dispersion in an elastic plate with microstructure: effects of characteristic length in a couple stress model*, Meccanica, **49**, 1083–1090.

26. Sharma V., Kumar S. (2017), *Effects of microstructure and liquid loading on velocity dispersion of leaky Rayleigh waves at liquid-solid interface*, Canadian Journal of Physics, doi: https://doi.org/10.1139/cjp-2016-0343.

27. Tanuma K., Man C.S., Chen Y. (2015), *Dispersion of Rayleigh waves in weakly anisotropic media with vertically-inhomogeneous initial stress*, International Journal of Engineering Science, **92**, 63–82.

28. Toupin R.A. (1962), *Elastic materials with couple-stresses*, Archive for Rational Mechanics and Analysis, **11**, 385–414.

29. Vavva M.G., Gergidis L.N., Protopappas V.C., Charalambopoulos A., Polyzos D., Fotiadis D.I. (2014), *A study on Rayleigh wave dispersion in bone according to Mindlin's Form II gradient elasticity*, Journal of the Acoustical Society of America, **135**, 5, 3117–3126.

30. Vavva M.G., Protopappas V.C., Gergidis L.N., Charalambopoulos A., Fotiadis D.I., Polyzos D. (2009), *Velocity dispersion of guided waves propagating in a free gradient elastic plates: application to cortical bone*, Journal of the Acoustical Society of America, **125**, 5, 3414–3427.

31. Vinh P.C., Anh V.T.N, Thanh V.P. (2014), *Rayleigh waves in an isotropic elastic half-space coated by a thin isotropic elastic layer with smooth contact*, Wave Motion, **51**, 496–504.

32. Voigt W. (1887), *Theoretical studies on the elasticity relationships of crystals* [in German: *Theoretische Studien über die Elastizitätsverhältnisse der Kristalle*], Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, 34.

33. Yang F., Chong A.C.M., Lam D.C.C., Tong P. (2002), *Couple stress based strain gradient theory for elasticity*, International Journal of Solids and Structures, **39**, 2731–2743.

34. Zhu J., Popovics J.S., Schubert F. (2004), *Leaky Rayleigh and Scholte waves at the fluid–solid interface subjected to transient point loading*, Journal of Acoustical Society of America, **116**, 4, 2101–2110.