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UNIFICATION OF VARIOUS TYPE EXPRESSIONS FOR PROBABILITY DISTRIBUTION OF ARBITRARY RANDOM NOISE AND VIBRATION WAVES BASED ON THEIR ACTUAL FLUCTUATION RANGES (THEORY AND EXPERIMENT)

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In the measurement of actual random phenomena, the observed data often result in a loss or a distortion of information due to the existence of a definite dynamic range of measurement equipments. In this paper, a unified expression of the fluctuation probability distribution for an environmental noise or vibration wave is proposed in an actual case when this wave has a finite range of amplitude fluctuation in itself or is measured through the usual instruments (e.g., sound level meter, level recorde, etc.) with a finite·dynamic range. The resultant expression of the probability distribution function has been derived in a form of the statistical Jacobi series type expansion taking a Beta distribution as the 1st expansion term and Jacobi polynomial as the orthogonal polynomial. This unified probability expression contains the well-known statistical Gegenbauer series type probability expansion as a special case, and the statistical Laguerre and Hermite series type probability expansions as two special limiting cases. Finally, the validity of the proposed theory has been experimentaly confirmed by applying to the actually observed data of a road traffic noise. This statistical Jacobi series expression shows good agreement with experimentally sampled points as compared with other types of statistical series expression.

1. Introduction

It is well-known that the Gaussian distribution is of essential importance as a standard type probability distribution expression of random noise or vibration, not only in the outdoor but also in the indoor acoustics. Moreover, not inferior to the Gaussian distribution, as a probability distribution for the fluctuation of the sound intensity – for instance in the diffused sound field, the Gamma distribution plays an important role, as seen in the studies by WATERHOUSE [1], LUBMAN [2] and so forth [3, 4]. For the road traffic noise, the Gamma distribution is widely known as the distribution of sound intensity, the sound energy density or the distance between two vehicles (involving the Erlang distribution [5] as a special case). In general, the Gamma distribution can be employed as the first order approximation to the arbitrary type probability distribution expression of a positive random variable. However, the field of environmental noise or vibration seems to be more complex due to the physical, social and human psychological causes. So, in such a field, the observed random wave shows a complicated fluctuation pattern with arbitrary distribution forms apart from a usual Gaussian or Gamma distribution. Furthermore, in the actual observation, it very often happens that the recorded pattern of an objective random wave has a limited fluctuation amplitude domain owing to the dynamic range of measurement device – its probability distribution has to change naturally in its functional shape from an original one with non constrained amplitude fluctuation.

As is well-known, as the general type probability distribution expression for non-Gaussian and non-Gamma variables, the statistical Hermite expansion series type expression [6] including the well-known Gram-Charlier A type expansion [7] defined within a fluctuation domain $(-\infty, \infty)$ and taking the Gaussian distribution as the first expansion term, and the Laguerre one within a fluctuating domain $[0, \infty]$ and taking the Gamma distribution as the first expansion term have been not only theoretically proposed but also frequently applied to the indoor or outdoor actual sound or vibration environment. For example, as reported [1], when the input wave of sound pressure or vibration acceleration with the statistical Hermite type arbitrary distribution passes through the mean-squared circuit of sound level meter or vibration meter, the fluctuation of its output response wave can be described by the statistical Laguerre expansion series type probability expression.

The fluctuating amplitude of the actual phenomenon can not take every value within all parts of the theoretically defined ideal range $(-\infty, \infty)$ or $[0, \infty)$ but has some kind of limited fluctuation range. Additionally, its fluctuation range is usually constrained by the existence of dynamic range of measuring equipments. Therefore, when only the above two kinds of statistical Hermite or Laguerre type orthonormal expansion series type expressions are applied to the actual situations, there remains some discrepancy between theory and experiment, especially at the tips of the fluctuation amplitude (for example, corresponding to evaluation indices L_5, L_{10}, L_{95}, L_{90} and so on). And, in case of using the theoretical expansion type expressions with no matching to the objective phenomena, many expansion terms had to be introduced for the purpose of reflecting the higher order moments directly connected with the tips of the fluctuation. From the above practical points of view, the limitation of amplitude should be actively introduced into the present theoretical consideration from the starting point of study and calls for a new distribution expression which involves the above two kinds of Hermite and Laguerre type probability expressions as two special cases.

In this paper, in order to obtain a better adaptation to the limitation of amplitude fluctuation and a wider application, it is more important in the actual sound or vibration environment to grasp more correctly the diversified fluctuation distribution forms of input, before considering the effect of various type characteristics of sound system itself. Thus, this study is first on how to find a new kind of unified probability distribution expression of the random fluctuation appearing in the complicated sound or vibration environment. More concretely, the unified probability density function is newly derived in a form of statistical Jacobi expansion series type expression (including the Gegenbauer expansion series type expression) taking the well-known Beta distribution as the first expansion term and the higher order moments as the Jacobi polynomial type statistics in each expansion coefficient. Finally, the effectiveness of the proposed method is experimentally confirmed by applying it to the digital simulation data and to the actually observed road traffic noise data.

2. Theoretical consideration

2.1. Probability density expression with the limitation of amplitude fluctuation range

2.1.1. A probability density expression in a unified form

Statistical Jacobi expansion series type. Now, an arbitrary variable X fluctuating only within a finite interval $[a, b]$ is taken into consideration. It is necessary to normalize the fluctuation range so as to investigate generally the influence of the limitation of the fluctuation amplitude on the resultant probability expression in a unified form. Hereupon, let us first pay attention to the variable fluctuating within the interval $[0, 1]$. Introducing an arbitrary weighting function $p(x)$ defined within the interval and the orthonormal polynomials based on $p(x)$, a probability density function (abbr., p.d.f) $P(x)$ can be expressed by the following distribution expansion

$$P(x) = \sum_{n=0}^{\infty} A_n p(x) \Phi_n^*(x), \tag{1}$$

where $\{\Phi_n^*(x)\}$ forms a complete set of orthonormal function with respect to $p(x)$

$$\int_0^1 \Phi_m^*(x) \Phi_n^*(x) p(x) dx = \delta_{mn}. \tag{2}$$

Then, by use of this relationship, every expansion coefficient can be immediately calculated in the following:

$$A_n \triangleq \langle \Phi_n^*(x) \rangle = \int_0^1 \Phi_n^*(x) P(x) dx, \tag{3}$$

where $\langle \cdot \rangle$ denotes the statistical averaging operation with respect to the distribution P. The weighting function $p(x)$ can be set arbitrarily in advance so as to satisfy the following fundamental properties of p.d.f.

$$\int_0^1 p(x)\,dx = 1. \tag{4}$$

Since weighting function defined within [0, 1], the well-known Beta distribution can be reasonably chosen,

$$p(x) = \frac{1}{B(\gamma, \alpha-\gamma+1)} x^{\gamma-1}(1-x)^{\alpha-\gamma}, \tag{5}$$

$$B(p,q) \triangleq \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \tag{6}$$

Then, from Eq. (2), the orthonormal function can be determined as

$$\Phi_n^*(x) = \sqrt{\frac{(\alpha+2n)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\alpha+n)\,\Gamma(\gamma+n)}{\Gamma(\alpha+1)\,n!\,\Gamma(n+\alpha-\gamma+1)\,\Gamma(\gamma)}}\,G_n(\alpha,\gamma;x), \tag{7}$$

where $\Gamma(z)$ is the Gamma function and $G_n(\alpha,\gamma;x)$ is the Jacobi polynomial defined by:

$$G_n(\alpha,\gamma;x) \triangleq \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} x^{1-\gamma}(1-x)^{\gamma-\alpha}\left(\frac{d}{dx}\right)^n [x^{\gamma+n-1}(1-x)^{\alpha+n-\gamma}]. \tag{8}$$

The employment of Jacobi polynomial at the beginning of analysis brings some generality to the resultant expressions. That is, this Jacobi polynomial coincides with the other types of orthogonal polynomials, e.g., Gegenbauer polynomial, Legendre polynomial, Tchebycheff polynomial, etc., when α and γ take certain particular values. Consequently, substituting Eqs. (5) and (7) into Eq. (1), the p.d.f. for x is derived in a general form of the expansion series as:

$$P(x) = \frac{1}{B(\gamma,\alpha-\gamma+1)} x^{\gamma-1}(1-x)^{\alpha-\gamma}\left\{1+\sum_{n=1}^{\infty} A_n \sqrt{\frac{(\alpha+2n)\Gamma(\alpha-\gamma+1)\Gamma(\alpha+n)\Gamma(\gamma+n)}{\Gamma(\alpha+1)n!\Gamma(n+\alpha-\gamma+1)\Gamma(\gamma)}} \times G_n(\alpha,\gamma;x)\right\}. \tag{9}$$

From Eq. (3), the expansion coefficient A_n in the above expression can be immediately calculated by using from the 1st to the nth order moment statistics

$$A_n \triangleq \langle \Phi_n^*(x) \rangle = \sqrt{\frac{(\alpha+2n)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\alpha+n)\,\Gamma(\gamma+n)}{\Gamma(\alpha+1)\,n!\,\Gamma(n+\alpha-\gamma+1)\,\Gamma(\gamma)}}\langle G_n(\alpha,\gamma;x)\rangle. \tag{10}$$

Thus, the p.d.f. of the variable x confined within a finite fluctuation range [0, 1] is expressed by the orthonormal series with the Beta distribution being as the first term of expansion and with expansion coefficients which can be estimated through the statistics in the form of the empirical averages of consecutive Jacobi polynomials.

The use of Jacobi polynomials having two parameters α and γ is advantageous in expressing roughly the shape of the p.d.f. by lower order approximations because it is fundamentally important to catch first the statistical information on two lower order moments like the mean and variance. The dominant part of the expansion (9) can be reflected by few first terms due to the proper adjustment of these two parameters α and γ. It is easy to transform the above representation of p.d.f. with a normalized fluctuation interval [0, 1] to the general case with an arbitrary fluctuation interval $[a, b]$ by means of a transformation of variables

$$x \triangleq (X-a)/(b-a). \tag{11}$$

Namely, the p.d.f. $P_x(X)$ of the actual variable X can be directly derived by using the measure-preserving transformation of probability:

$$P_x(X) = P(x)\left|\frac{dx}{dX}\right|\Big|_{x=(X-a)/(b-a)} = \sum_{n=0}^{\infty} A_n P_0(X)\,\Phi_n(X), \tag{12}$$

where

$$P_0(X) = \frac{1}{B(\gamma, \alpha-\gamma+1)(b-a)^{\alpha}}(X-a)^{\gamma-1}(b-X)^{\alpha-\gamma}, \tag{13–a}$$

$$\begin{aligned}\Phi_n(X) &= \Phi_n^*(x)\big|_{x=(X-a)/(b-a)} \\ &= \sqrt{\frac{(\alpha+2n)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\alpha+n)\,\Gamma(\gamma+n)}{\Gamma(\alpha+1)n!\,\Gamma(n+\alpha-\gamma+1)\,\Gamma(\gamma)}}\,G_n\!\left(\alpha,\gamma;\frac{X-a}{b-a}\right),\end{aligned} \tag{13–b}$$

$$A_n = \sqrt{\frac{(\alpha+2n)\,\Gamma(\alpha-\gamma+1)\,\Gamma(\alpha+n)\,\Gamma(\gamma+n)}{\Gamma(\alpha+1)n!\,\Gamma(n+\alpha-\gamma+1)\,\Gamma(\gamma)}}\left\langle G_n\!\left(\alpha,\gamma;\frac{X-a}{b-a}\right)\right\rangle. \tag{13–c}$$

Here, it may be advantageous to impose the additional conditions $A_1 = A_2 = 0$ upon the expansion coefficients. These constants determine α and γ uniquely. The unknown parameters α, γ resulting from the above assumption can be explicitly determined by means of mean μ_x and variance σ_x^2 of the actually observed variable X, respectively, as follows (i.e. moment method):

$$\left.\begin{aligned}\alpha &= \frac{(\mu_x-a)(b-\mu_x)}{\sigma_x^2}-2, \\ \gamma &= \frac{\mu_x-a}{b-a}\left\{\frac{(\mu_x-a)(b-\mu_x)}{\sigma_x^2}-1\right\}.\end{aligned}\right\} \tag{14}$$

2.1.2 A probability density expression in a symmetrical form

Statistical Gegenbauer expansion series type. Let us consider a special case of choosing a p.d.f. symmetrical with respect to the center $(a+b)/2$ of the amplitude fluctuation interval. According to the above symmetrical property, two parameters α,

γ must satisfy first the following relation

$$\alpha = 2\gamma - 1. \tag{15}$$

In order to investigate the effect of the symmetry on the resultant probability distribution form, let us introduce the following parameter

$$\nu = \gamma - 1/2\,(\nu > -1/2). \tag{16}$$

[A] Eq. (13–a) as the first term of expansion series type expression in Eq. (12) can be transformed, as follows:

$$\begin{aligned} P_0(X) &= \frac{1}{B(\nu+1/2, \nu+1/2)(b-a)}\left(\frac{X-a}{b-a}\right)^{\nu-1/2}\left(1-\frac{X-a}{b-a}\right)^{\nu-1/2} \\ &= \frac{1}{B(\nu+1/2, \nu+1/2)(b-a)/2\cdot 2^{2\nu}}\left[1-\left\{\frac{X-(a+b)/2}{(b-a)/2}\right\}^2\right]^{\nu-1/2}. \end{aligned} \tag{17}$$

Then, the duplication formula of Gamma function [8]:

$$\Gamma(2Y) = \frac{2^{2Y}}{2\sqrt{\pi}}\Gamma(Y)\Gamma(Y+1/2) \tag{18}$$

and equality $\Gamma(1/2) = \sqrt{\pi}$ give

$$B(\nu+1/2, \nu+1/2)2^{2\nu} = B(\nu+1/2, 1/2). \tag{19}$$

Substituting Eq. (19) into Eq. (17) leads to:

$$P_0(X) = \frac{1}{B(\nu+1/2, 1/2)(b-a)/2}\left[1-\left\{\frac{X-(a+b)/2}{(b-a)/2}\right\}^2\right]^{\nu-1/2}. \tag{20}$$

[B] In the same way, Eq. (13–b) can be transformed by use of the above symmetrical property, and Eqs. (15) and (16).

$$\begin{aligned} \Phi_n(X) &= \sqrt{\frac{(2\nu+2n)\,\Gamma(\nu+1/2)\,\Gamma(2\nu+n)\,\Gamma(\nu+n+1/2)}{\Gamma(2\nu+1)n!\,\Gamma(n+\nu+1)\,\Gamma(\nu+1/2)}}\,G_n\left(2\nu, \nu+1/2; \frac{X-a}{b-a}\right) \\ &= (-1)^n\sqrt{\frac{2(\nu+n)n!\,[\Gamma(2\nu)]^2}{\Gamma(2\nu+1)\,\Gamma(2\nu+n)}}(-1)^n\frac{\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)}\times \\ &\quad \times G_n\left(2\nu, \nu+1/2;\ \frac{1}{2}\left\{1+\frac{X-(a+b)/2}{(b-a)/2}\right\}\right). \end{aligned} \tag{21}$$

Then, based on a relationship:

$$\frac{2[\Gamma(2\nu)]^2}{\Gamma(2\nu+1)} = B(\nu+1/2, 1/2)\frac{2^{2\nu-1}[\Gamma(\nu)]^2}{\pi}, \tag{22}$$

Eq. (21) can be rewritten as

$$\Phi_n(X) = (-1)^n \sqrt{B(\nu+1/2, 1/2)\frac{(\nu+n)n!\,2^{2\nu-1}[\Gamma(\nu)]^2}{\pi\,\Gamma(2\nu+n)}}$$

$$\times(-1)^n \frac{\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)} G_n\left(2\nu, \nu+1/2; \frac{1}{2}\left\{1+\frac{X-(a+b)/2}{(b-a)/2}\right\}\right). \quad (23)$$

In the sequel, by virtue of the relationship between Jacobi and Gegenbauer polynomials

$$(-1)^n \frac{\Gamma(n+2\nu)}{n!\,\Gamma(2\nu)} G_n\left(2\nu, \nu+1/2; \frac{1+x}{2}\right) = C_n^\nu(x), \quad (24)$$

the above $\Phi_n(X)$ can be finally expressed as follows:

$$\Phi_n(X) = (-1)^n \sqrt{B(\nu+1/2, 1/2)\frac{(\nu+n)n!\,2^{2\nu-1}[\Gamma(\nu)]^2}{\pi(2\nu+n)}}\, C_n^\nu\left(\frac{X-(a+b)/2}{(b-a)/2}\right), \quad (25)$$

where $C_n^\nu(X)$ is the Gegenbauer polynomial.

[C] The expansion coefficient A_n can also be immediately calculated by the statistics:

$$A_n \triangleq \langle\Phi_n(X)\rangle = (-1)^n \sqrt{\frac{B(\nu+1/2, 1/2)(\nu+n)n!\,2^{2\nu-1}[\Gamma(\nu)]^2}{\pi\Gamma(n+2\nu)}} \left\langle C_n^\nu\left(\frac{X-(a+b)/2}{(b-a)/2}\right)\right\rangle. \quad (26)$$

Thus, the p.d.f. of the random variable X with an arbitrary fluctuation amplitude $[a, b]$ is given by the following expression in the form of the statistical Gegenbauer expansion series:

$$P(X) = \frac{1}{B(\nu+1/2, 1/2)(b-a)/2}\left[1-\left\{\frac{X-(a+b)/2}{(b-a)/2}\right\}^2\right]^{\nu-1/2} \left\{1+\sum_{n=1}^{\infty} A_n \times\right.$$

$$\left.\times(-1)^n \sqrt{\frac{B(\nu+1/2, 1/2)(\nu+n)n!\,2^{2\nu-1}[\Gamma(\nu)]^2}{\pi\,\Gamma(2\nu+n)}}\, C_n^\nu\left(\frac{X-(a+b)/2}{(b-a)/2}\right)\right\}. \quad (27)$$

And, from Eq. (14), the mean and the variance are expressed respectively as follows:

$$\mu_x = (a+b)/2, \quad \sigma_x^2 = \left(\frac{b-a}{2}\right)^2 \frac{1}{2(\nu+1)}. \quad (28)$$

2.2. Probability density expression with an infinite amplitude fluctuation range

Connection with the well-known generalized p.d.f. expressions. As is well-known, the standard Gaussian distribution with amplitude domain $(-\infty, \infty)$ has been applied to many kinds of statistical problems on the random waves. With the same fluctuation domain, the statistical Hermite expansion series type expression (for instance, the Gram-Charlier A type of expansion expression) has been used to express an arbitrary distribution expression of non-Gaussian type. In the field of environmental noise or vibration, the above generalized distribution expression and approximately its first expansion term, the Gaussian distribution, have been both very frequently used to describe the p.d.f. of the sound pressure, vibration acceleration and/or level fluctuation. On the other hand, there is a generalized statistical Laguerre expansion series type expression for the random waves fluctuating only within a positive interval $[0, \infty)$ (e.g., the energy fluctuation in the same field), whose first term of expansion is the Gamma distribution. Even though it seems apparently true that the infinite domain $(-\infty, \infty)$ involves domain $[0, \infty)$ and a finite one $[a, b]$, the distribution expression, which is defined within the domain $[a, b]$ and reflects much more strictness of restrictions in theoretical analysis, can contain the probability expression defined within $(-\infty, \infty)$ or $[0, \infty)$ as two special cases where the restrictions are loosened. Therefore, the probability density expression in Eqs. (12), (13) or (27) must agree as special cases with the statistical Hermite and statistical Laguerre expansion series type expressions previously reported. Through such a theoretical consideration, the validity of the proposed expression can be shown within the theoretical extent as follows:

2.2.1. Relation to the statistical Hermite series expansion type expression. First, let us consider the usual case when the random variable originally fluctuates freely in both positive and negative intervals $(-\infty, \infty)$ under no constraint of amplitude limitation, and first focus on the statistical Gegenbauer series type expression of Eq. (27) having a symmetrical property with respect to the center of the fluctuation domain.

By solving Eq. (28) with respect to a and b, the following relationship can be derived.

$$\begin{aligned} a &= \mu_\alpha - \sqrt{2(\nu+1)}\,\sigma_x, \\ b &= \mu_x + \sqrt{2(\nu+1)}\,\sigma_x. \end{aligned} \tag{29}$$

Substituting Eq. (29) into Eqs. (20), (25) and (26), the first expansion term $P_0(X)$, the orthogonal polynomial $\Phi_n(X)$ and the expansion coefficient A_n could be rewritten respectively as follows:

$$P_0(X) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \frac{\nu\Gamma(\nu)}{\Gamma(\nu+1/2)\sqrt{\nu+1}} \left[1 - \frac{(X-\mu_x)^2/2\sigma_x^2}{\nu+1} \right]^{\nu+1-3/2}, \tag{30–a}$$

$$\Phi_n(X) = (-1)^n \sqrt{B(\nu+1/2, 1/2) \frac{(\nu+n)n!\,2^{2\nu-1}[\Gamma(\nu)]^2}{\pi\Gamma(2\nu+n)}}\, C_n^\nu\!\left(\frac{X-\mu_x}{\sqrt{2(\nu+1)}\sigma_x} \right), \tag{30–b}$$

$$A_n = (-1)^n \sqrt{B(\nu+1/2, 1/2) \frac{(\nu+n)n!\,2^{2\nu-1}[\Gamma(\nu)]^2}{\pi\Gamma(2\nu+n)}} \left\langle C_n^\nu\!\left(\frac{X-\mu_x}{\sqrt{2(\nu+1)}\,\sigma_x} \right) \right\rangle \tag{30–c}$$

Here, the limiting case with no amplitude constraint: $a \to -\infty$ and $b \to \infty$ can be obtained in the case when $\nu \to \infty$ (from Eq. (29)).

[A] First of all, let us consider $P_0.(X)$. From the Stirling's formula [8]:

$$\Gamma(\nu) \sim \sqrt{2\pi e^{-\nu} \nu^{\nu-1/2}} \, (\nu \to \infty), \tag{31}$$

the following relationship can be derived.

$$\frac{\nu\Gamma(\nu)}{\Gamma(\nu+1/2)\sqrt{\nu+1}} \sim 1 \quad (\nu \to \infty). \tag{32}$$

Furthermore, the well-known property $(1+1/y)^y \to e \; (y \to \infty)$ gives

$$\left[1 - \frac{(X-\mu_x)^2/2\sigma_x^2}{\nu+1}\right]^{\nu+1-3/2} \to e^{-(x-\mu_x)^2/2\sigma_x^2} \, (\nu \to \infty). \tag{33}$$

When $\nu \to \infty$, the first expansion term $P_0(X)$ asymptotically approaches to the Gaussian distribution with mean μ_x and variance σ_x^2.

$$P_0(X) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-\mu_x)^2/2\sigma_x^2}. \tag{34}$$

[B] $\Phi_n(X)$ in Eq. (30–b) can be rewritten as follows:

$$\Phi_n(X) = \frac{(-1)^n}{\sqrt{n!}} \sqrt{\frac{\Gamma(\nu+1/2)\sqrt{\pi}(\nu+n)\,2^{2\nu-1}\,[\Gamma(\nu)]^2}{\Gamma(\nu+1)\,\pi\,\Gamma(n+2\nu)(2\nu)^{-n}}\, n!\,(2\nu)^{-n/2}\, C_n^\nu\!\left(\frac{X-\mu_x}{\sqrt{2(\nu+1)}\sigma_x}\right)}. \tag{35}$$

Using the relationship:

$$\frac{\Gamma(\nu+1/2)\sqrt{\pi}(\nu+n)\,2^{2\nu-1}\,[\Gamma(\nu)]^2}{\Gamma(\nu+1)\,\pi\,\Gamma(n+2\nu)(2\nu)^{-n}} \to 1 \, (\nu \to \infty) \tag{36}$$

and considering the limiting property related to Hermite polynomial $H_n(\cdot)$:

$$n!(2\nu)^{-n/2}\, C_n^\nu\!\left(\frac{X-\mu_x}{\sqrt{2(\nu+1)}\,\sigma_x}\right) \to H_n\!\left(\frac{X-\mu_x}{\sigma_x}\right) (\nu \to \infty), \tag{37}$$

the orthonormal polynomial $\Phi_n(X)$ in Eq. (35) asymptotically changes as:

$$\Phi_n(X) = \frac{(-1)^n}{\sqrt{n!}} H_n\!\left(\frac{X-\mu_x}{\sigma_x}\right). \tag{38}$$

[C] Through the same asymptotical procedure as in Eq. (37), the expansion coefficient can be also rewritten as follows:

$$A_n = \frac{(-1)^n}{\sqrt{n!}} \left\langle H_n\!\left(\frac{X-\mu_x}{\sigma_x}\right)\right\rangle. \tag{39}$$

Therefore, by use of the asymptotical relationships of Eqs. (34), (38) and (39), it could be proved that the statistical Gegenbauer expansion series type expression completely agrees with the well-known statistical Hermite expansion series type expression defined within $(-\infty, \infty)$. From the proof process, it can be easily noticed that if the values of μ_x and σ_x^2 are employed as arbitrary parameters in advance, Eq. (14), of course, is not necessary to be satisfied (when μ_x and σ_x^2 are adopted as the mean and the variance of X respectively under the conditions $A_1 = A_2 = 0$, the statistical Hermite expansion series type expression could coincide with the Gram-Charlier A series type expression).

2.2.2. Relation to the statistical Laguerre series expansion type expression. By introducing two kinds of new parameters m and s satisfying the relationships: $\gamma = m$, $a = 0$, $b = \alpha s$, Eqs. (13–a), (13–b) and (13–c) can be rewritten as:

$$P_0(X) = \frac{1}{B(m, \alpha - m + 1)\, \alpha s} \left(\frac{X}{\alpha s}\right)^{m-1} \left(1 - \frac{X}{\alpha s}\right)^{\alpha - m}, \tag{40–a}$$

$$\Phi_n(X) = \sqrt{\frac{\Gamma(\alpha - m + 1)(\alpha + 2n)\, \Gamma(\alpha + n)\, \Gamma(m + n)}{\Gamma(\alpha + 1)\, n!\, \Gamma(n + \alpha - m + 1)\, \Gamma(m)}}\, G_n(\alpha, m; X/\alpha s), \tag{40–b}$$

$$A_n = \sqrt{\frac{\Gamma(\alpha - m + 1)(\alpha + 2n)\, \Gamma(\alpha + n)\, \Gamma(m + n)}{\Gamma(\alpha + 1)\, n!\, \Gamma(n + \alpha - m + 1)\, \Gamma(m)}} \left\langle G_n(\alpha, m; X/\alpha s) \right\rangle. \tag{40–c}$$

[A] First, we pay our attention to $P_0(X)$ and rewrite Eq. (40–a) as:

$$P_0(X) = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - m + 1)\, \alpha^m} \frac{X^{m-1}}{\Gamma(m)\, s^m} \left(1 - \frac{X}{\alpha s}\right)^{\alpha - m}. \tag{41}$$

By using the Stirling's formula, the coefficient in Eq. (41) becomes

$$\frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - m + 1)\, \alpha^m} \sim 1. \tag{42}$$

Therefore, the first term of expansion in Eq. (12) approaches asymptotically to the well-known Gamma distribution when $\alpha \to \infty$

$$P_0(X) = \frac{X^{m-1}}{\Gamma(m)\, s^m} e^{-X/s}. \tag{43}$$

[B] Secondly, Eq. (40–b) can be rewritten as

$$\Phi_n(X) = \sqrt{\frac{\Gamma(m)\, n!\, \Gamma(\alpha - m + 1)(\alpha + 2n)\, \Gamma(\alpha + n)}{\Gamma(m + n)\, \Gamma(\alpha + 1)\, \Gamma(n + \alpha - m + 1)}}\, \frac{\Gamma(m + n)}{n!\, \Gamma(m)}\, G_n(\alpha, m; X/\alpha s). \tag{44}$$

Considering the asymptotical relationship:

$$\frac{\Gamma(\alpha - m + 1)(\alpha + 2n)\, \Gamma(\alpha + n)}{\Gamma(\alpha + 1)\, \Gamma(n + \alpha - m + 1)} \to 1 \, (\alpha \to \infty) \tag{45}$$

and the limiting property related to a Laguerre polynomial $L_n^{(m-1)}(\cdot)$:

$$\frac{\Gamma(m+n)}{n!\,\Gamma(m)} G_n(\alpha, m; X/\alpha s) \to L_n^{(m-1)}(X/s)\,(\alpha \to \infty), \tag{46}$$

Eq. (44) becomes

$$\Phi_n(X) = \sqrt{\frac{\Gamma(m)n!}{\Gamma(m+n)}}\, L_n^{(m-1)}(X/s). \tag{47}$$

[C] Through the same asymptotical procedure as in the previous orthonormal polynomial, the expansion coefficient can be also rewritten as follows when $\alpha \to \infty$.

$$A_n = \sqrt{\frac{\Gamma(m)n!}{\Gamma(m+n)}} \langle L_n^{(m-1)}(X/s) \rangle. \tag{48}$$

Therefore, the statistical Jacobi expansion series type expression in Eq. (12) agrees completely with the well-known statistical Laguerre one in the asymptotic case when $\alpha \to \infty$.

3. Experimental consideration

In the actual field of noise or vibration the statistical Hermite and Laguerre expansion series type expression are very often used as the p.d.f. for many kinds of random fluctuations, e.g. the road traffic noise or vibration, the sound pressure level and the sound intensity in a room. Accordingly, many results reported in the previous papers by applying the statistical Hermite and Laguerre expansion series type expressions to the actual environment can also be recognized as the experimental confirmation of the proposed method. In this paper, through the experimental results on the application to the actual data, it has been confirmed that the statistical Jacobi expansion series type expression is best matched to the actual phenomena with constraint of their fluctuating amplitude limitation. First of all, the effectiveness of the statistical Gegenbauer expansion series type expression has been confirmed by means of the digital simulation technique. Then, the statistical Jacobi expansion series type expression has been applied to the actually observed road traffic noise.

3.1. Digital simulation

In this section, the proposed Gegenbauer expansion series type expression (a special case of the Jacobi expansion series type expression) is experimentally compared with the well-known statistical Hermite expansion series type expression.

[i] Establishment of random model (a distribution with axial symmetry)

In order to investigate how the limitation of fluctuating amplitude interval affects the resultant p.d.f., the saturating non-linear system

$$X=\begin{cases}\frac{s}{A}U(|U|<A),\\ s\cdot \mathrm{sgn}\, U(|U|\geqslant A),\end{cases} \tag{49}$$

is adopted as the random where U is an idealized "true" value and X is the observed model. Concretely, the nonlinear property of this system is changed by selecting a threshold value A as 0.25, 1.0 and so on.

As a random system input U, the following four cases have been adopted:

Case A normal random numbers (mean = 0, variance = 1);

Case B (as a model reflecting the actual situation to some extent) a sum of normal random numbers (mean = 0, variance = 0.25) and a sine wave with amplitude 1 and frequency 10;

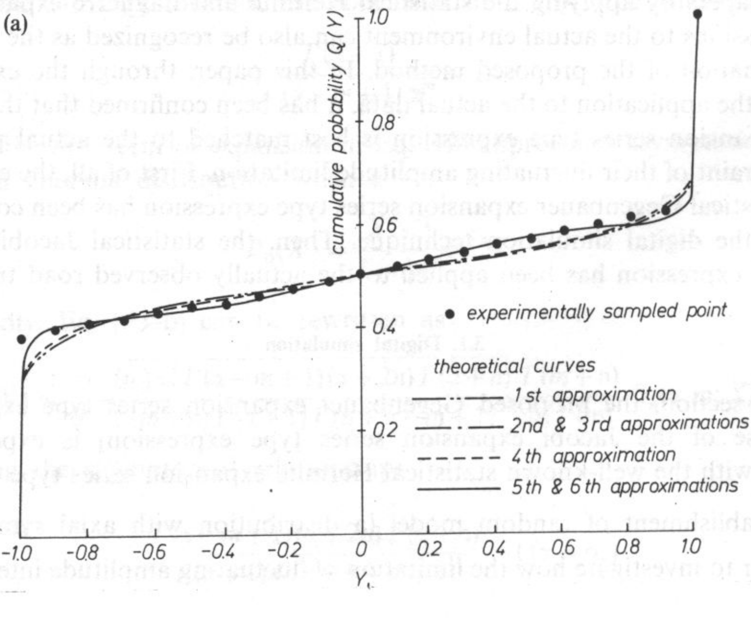
Case C (as an extreme example of non-Gaussian input) random numbers uniformly distributed within $[-2, 2]$;

Case D the same random input as in Case C and a sine wave with amplitude 1 and frequency 10.

[ii] Experimental results and discussions

Figure 1 (a) shows a comparison between the cumulative distribution function $Q(Y)=(\int_{\infty}^{Y}P(Y)dY)$ of Gegenbauer expansion series type expression and experimentally sampled poonts for Case D. The abscissa has been normalized by $Y=\{X-(a+b)/2\}/\{(b-a)/2\}$. The deviation $\varepsilon(Y)$ of the cumulative probability distribution from a Beta distribution is shown in Fig. 1 (b). From these figures, it is

(a)



obvious that the successive addition of higher order expansion terms moves theoretical curves close to the experimentally sampled points. Fig. 1 (c) shows the result of the Hermite expansion series type expression applied to the same experimental data. As for the saturated nonlinear phenomenon, these experimental results obviously clarify that the proposed Gegenbauer expansion series type

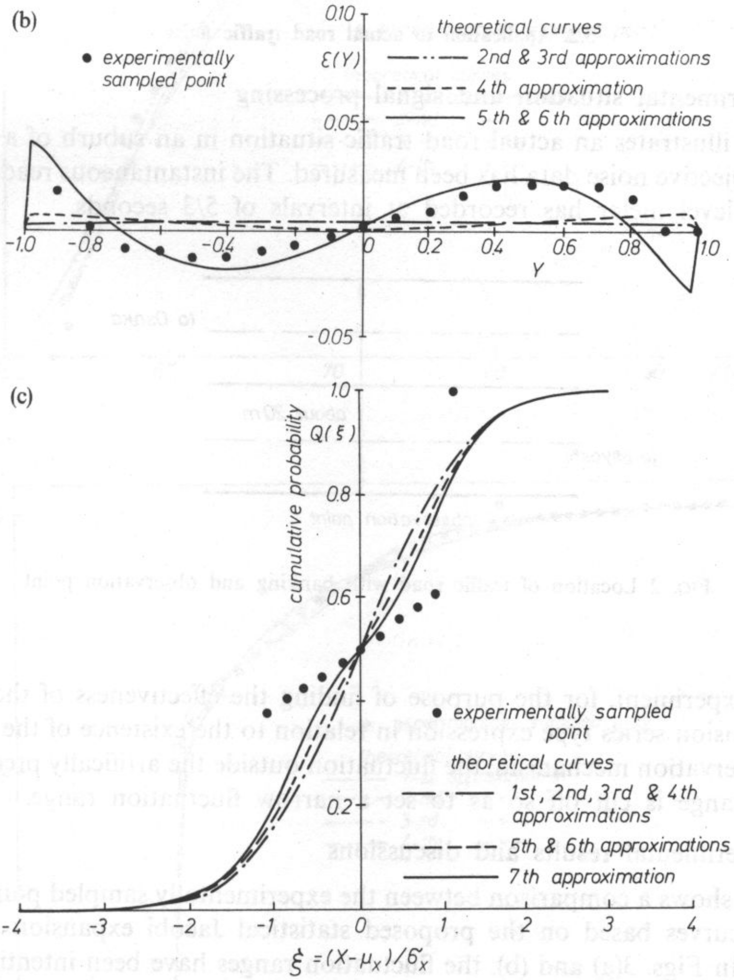


FIG. 1. A comparison between theory and experiment for a Case D(A = 0.25, ν = −0.395)

FIG. 1 (a) comparison between theoretical curves and experimentally sampled points for cumulative distribution function ($A_1 = 1.26 \times 10^{-2}$, $A_2 = 0.0$, $A_3 = 4.09 \times 10^{-1}$, $A_4 = -6.8$, $A_5 = -2.65 \times 10^{-1}$)

FIG. 1 (b) comparison between theoretical curves and experimentally sampled points for deviation $\varepsilon(Y)$ from the Beta distribution

FIG. 1 (c) comparison between theoretical curves of the statistical Hermite series expansion type and experimentally sampled points for cumulative probability distribution ($A_1 = A_2 = 0.0$, $A_3 = 1.8 \times 10^{-2}$, $A_4 = -7.7 \times 10^{-2}$, $A_5 = -6.4 \times 10^{-4}$, $A_6 = 1.9 \times 10^{-2}$)

expression has more flexibility necessary for applications than the well-used Hermite expansion series type expression.

Results for the other cases cited in [i] which have also given a good agreement between theory and experiment have been omitted.

3.2. Application to actual road traffic noise

[i] Experimental situation and signal processing

Figure 2 illustrates an actual road traffic situation in an suburb of a large city, where the objective noise data has been measured. The instantaneous reading (in dB) of a sound level meter has recorded at intervals of 5/3 seconds.

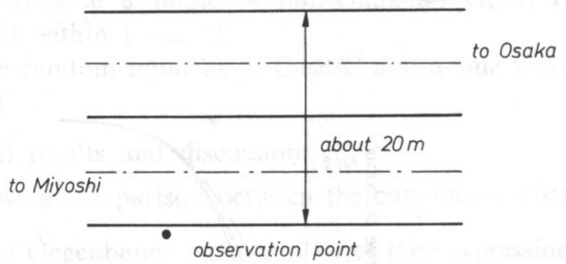


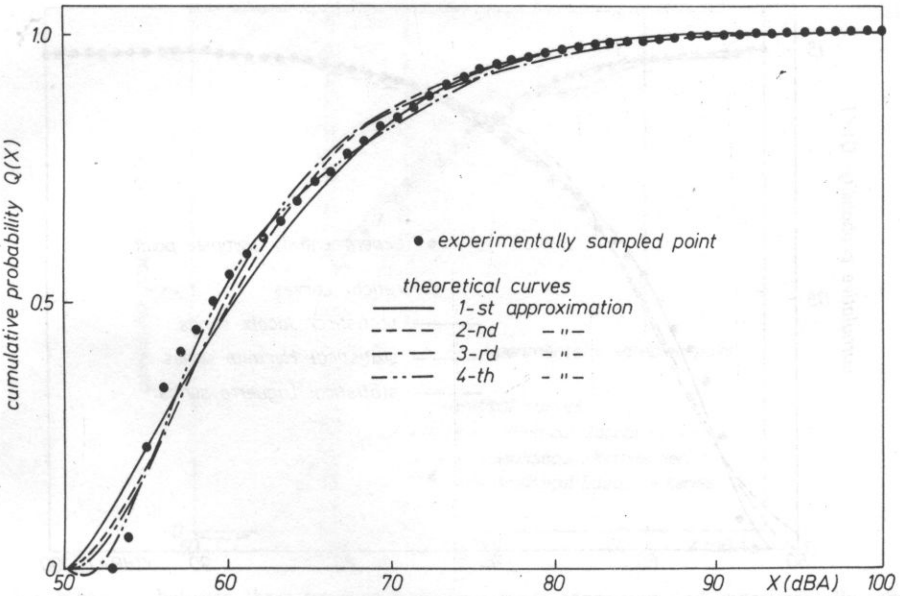
FIG. 2 Location of traffic road with banking and observation point

In this experiment, for the purpose of finding the effectiveness of the proposed Jacobi expansion series type expression in relation to the existence of the dynamical range of observation mechanism, the fluctuation outside the artifically preestablished dynamical range is cut off so as to set a narrow fluctuation range.

[ii] Experimental results and discussions

Figure 3 shows a comparison between the experimentally sampled points and the theoretical curves based on the proposed statistical Jacobi expansion series type expression. In Figs. 3(a) and (b), the fluctuation ranges have been intentionally and artificially set as [50, 100] and [60, 90] (dBA), respectively, to confirm the actual effectiveness of the proposed method.

Figure 4 shows a comparison between the experimentally sampled points and the theoretical curves based on the statistical Jacobi, Hermite and Laguerre expansion series type expressions. The fluctuation range has been purposely confined within [50, 100] (dBA) in advance and all of the theoretical curves have been drawn by employing the first 3 expansion terms. From Fig. 4, though the noise level completely fluctuates only within the preestablished range, it can be easily found that the



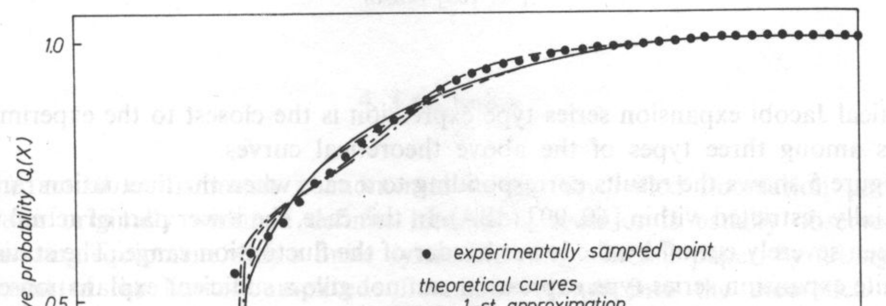


FIG. 3. A comparison between the experimentally sampled points and the theoretical curves based on the proposed statistical Jacobi expansion series type probability expression

FIG. 3(a) a case with a fluctuation domain [50, 100] (dBA)

FIG. 3(b) a case with a fluctuation domain [60, 90] (dBA)

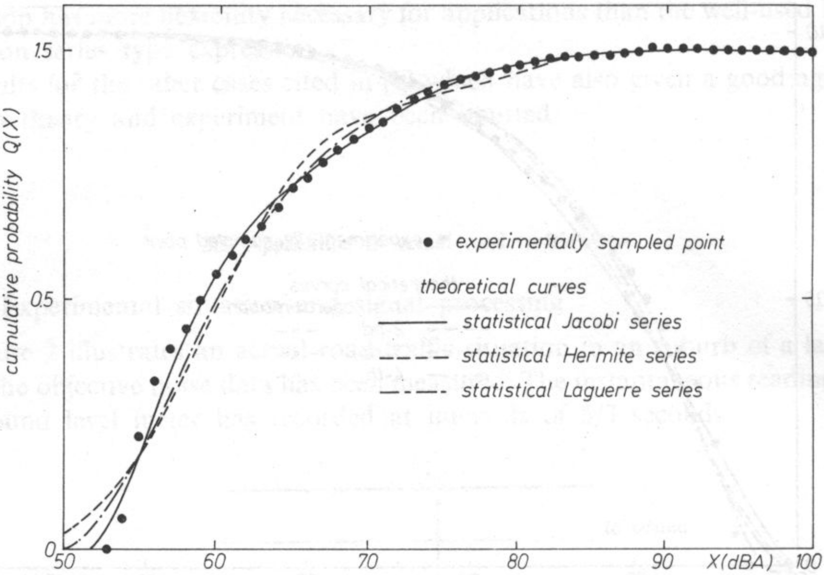


FIG. 4. A comparison between three types of series expansion expression and experimentally sampled points for cumulative probability distribution for an actual road traffic noise (the fluctuation domain is [50, 100] (dBA))

statistical Jacobi expansion series type expression is the closest to the experimental values among three types of the above theoretical curves.

Figure 5 shows the results corresponding to a case when the fluctuation range is artificially restricted within [60, 90] (dBA). In this case, the lower part of actual wave has been severely cut off by the lower border of the fluctuation range. The statistical Hermite expansion series type expression can not give a sufficient explanation of the experimental values and the statistical Laguerre expansion series type expression is diverged in accordance with considering the expansion terms of higher order. The newly proposed statistical Jacobi expansion series type expression provides us the best agreement with the experimental values not only in the vicinity of the borders but also in the central part of the distribution form.

The traffic volume in this observation has been small ($230 \sim 240$ vehicles/hour). As is well-known, it is generally difficult to predict the noise evaluation index in such a small traffic volume. Because the influence of the background noise could not be neglected, the cumulative level distribution has shown a rapid ascent and it is very difficult to make a sufficient approximation by means of only the usual Gaussian distribution. However, since the proposed unified expression involves originally the statistical Hermite and Laguerre expansion series type expressions as special cases, there is no doubt that the statistical Jacobi expansion series type expression can be sufficiently applied to the situation of large traffic volume too.

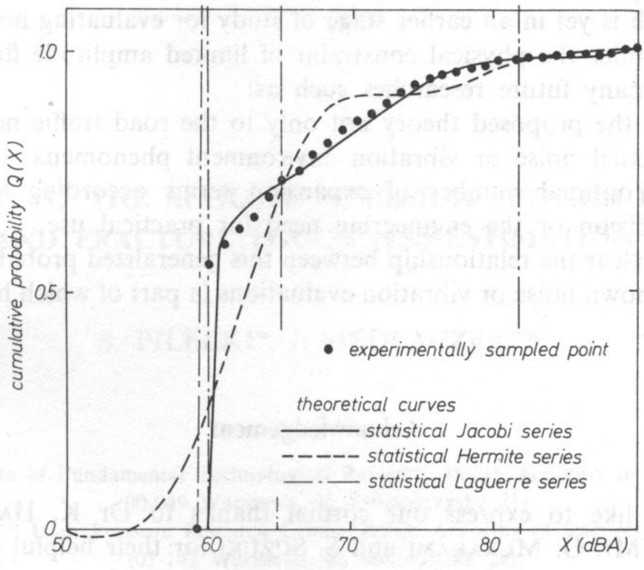


FIG. 5 A comparison between three types of series expansion expression and experimentally sampled points for cumulative probability distribution for an actual road traffic noise (the fluctuation domain is [60, 90] (dBA))

4. Conclusion

The actual environment phenomenon having diversified fluctuation patterns fluctuates originally within a definite interval in itself or is usually observed by a measuring equipment with a finite dynamical range. In this paper, by taking this constraint of the limited amplitude fluctuation range into the theoretical consideration, a statistical Jacobi expansion series type expression including a statistical Gegenbauer expansion series type expression has been newly derived as a unified probability distribution expression. It can be applied to many actual engineering fields as well as the field of noise or vibration, because it is a highly generalized distribution expression involving the statistical Hermite and Laguerre expansion series type expressions which take the well-known Gaussian and Gamma distribution as their first expansion term with their proper fluctuating amplitude range $(-\infty, \infty)$ and $[0, \infty)$.

To confirm the validity of the proposed theory, a digital simulation technique has been applied by employing the random numbers having axial symmetry. Then, the proposed theory has been also applied to the actual road traffic noise which can be poorly approximated by the ordinary Gaussian distribution in case of small traffic volume. The proposed theory has shown a good agreement with the experiment, in comparison with all the previously reported evaluation methods.

This reseach is yet in an earlier stage of study for evaluating noise or vibration environment under the physical constraint of limited amplitude fluctuation. Thus there remain many future researches such as:

1) to apply the proposed theory not only to the road traffic noise but also to many other actual noise or vibration environment phenomena;

2) to find optimal number of expansion terms according to the required prediction precision or the engineering need for practical use;

3) to make clear the relationship between this generalized probability expression and the well-known noise or vibration evaluations (a part of which has been already presented [9]).

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