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THE ACOUSTIC REACTANCE OF RADIATION OF A PLANAR ANNULAR MEMBRANE FOR AXIALLY-SYMMETRIC FREE VIBRATIONS

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The imaginary part of the acoustic impedance of radiation of a planar, annular membrane is analyzed. Axially-symmetric free vibrations, sinusoidally varying in time, are considered. An annular membrane is placed in a planar, rigid baffle and radiates an acoustic wave into a loss-less, homogeneous, gaseous medium. The reactance of radiation is obtained in an elementary form for high frequencies of radiated waves.

1. Introduction

The energetic aspect of radiating sound source is characterized most often through the acoustic power research or the impedance of radiation research. It concerns both — the real component and the imaginary component.

The theoretical analysis of acoustic impedance of radiation of a vibrating piston ring was undertaken by many authors (e.g. MERRIWEATHER [2], THOMPSON [5] and WYRZYKOWSKI [6]).

In [3] there are presented formulas for the real component of the acoustic power of radiation of a planar, annular membrane into a hemisphere filled with a loss-less, gaseous medium.

Referring to results produced in [3] in this work the imaginary component of the impedance of radiation of an annular membrane is considered. The membrane is placed in a planar, rigid baffle. Axially-symmetric, sinusoidally varying in time, free vibrations are analyzed. A starting point for this analysis is an integral formula written in a plane of complex variable for the power radiated by a planar, annular membrane. The imaginary part of this formula is integrated in case of high frequency of the radiated acoustic wave. An expression for the reactance of radiation has been obtained of an elementary form, useful for numerical calculations, and is illustrated graphically.

Results obtained in this work can be applied for active methods (cf. [4]) for the radiation of sound of vibrating surface sources research.

The analytical results are used for practical applications of membranes modeling (cf. [1]).

2. The analysis assumptions

Axially-symmetric free vibrations of a planar, annular membrane are

$$\frac{\eta_n(r)}{A_n} = J_0\left(x_n\frac{r}{r_1}\right) - \frac{J_0(x_n)}{N_0(x_n)}N_0\left(x_n\frac{r}{r_1}\right) \tag{2.1}$$

$r_1 < r < r_2$, where J_0, N_0 are cylindrical functions of null order correspondingly BESSEL'S and NEUMANN'S. The value x_n is the n-th eigenvalue of the frequency equation

$$\frac{J_0(kx_n)}{J_0(x_n)} = \frac{N_0(kx_n)}{N_0(x_n)},$$

where $k = r_2/r_1 \geq 1$. Also: $k_n = \omega_n\sqrt{\sigma/T}$, ω_n is the n-th eigenfrequency, corresponding to the mode $(0,n)$, σ – is the surface density of the membrane, T – is the stretching force of the membrane. Eigenvalues x_n were calculated for $n = 1, 2, ..., 6$, $k = 1.1, 1.2, 1.5, 2, 3$ and 5 and were placed in work [3].

Into a gaseous medium of rest density ϱ_0, propagation velocity of the acoustic wave c, averaged in time acoustic power [3] is radiated with frequency $\omega = k_0 c$, $k_0 = 2\pi/\lambda$.

$$N_n = N_n^{(\infty)}\frac{2\delta_n^2}{\alpha_n^2 - 1}\int\limits_0^\infty \frac{x}{\gamma}\left\{\frac{\alpha_n J_0(k\beta x) - J_0(\beta x)}{x^2 - \delta_n^2}\right\}^2 dx, \tag{2.2}$$

where $\delta_n = x_n/\beta$, $\beta = k_0 r_1$, $\alpha_n = J_0(x_n)/J_0(kx_n)$, $\gamma = \sqrt{1-x^2}$ for $x < 1$ and $\gamma = -i\sqrt{x^2-1}$ for $x > 1$. Also $N_n^{(\infty)} = \lim\limits_{k_0\to\infty} N_n$ and

$$N_n^{(\infty)} = \frac{1}{2}\varrho_0 c\int\limits_S v_n^2 dS, \qquad v_n = i\omega_n\eta_n(r),$$

where v_n is the vibration velocity of points of an annular membrane for the mode $(0,n)$.

Value $N_n/N_n^{(\infty)}$ is represents the normalized impedance of radiation $\zeta_n = \theta_n + i\chi_n$ of a planar, annular membrane for $(0,n)$ form of vibrations, θ_n is the normalized resistance (cf. [3]), χ_n is the normalized reactance.

3. The normalized radiation reactance

The normalized reactance of radiation $\chi_n = \Im\left(N_n/N_n^{(\infty)}\right)$ of a planar annular membrane for free vibrations $(0,n)$ have been derived from the formula (2.2)

$$\chi_n = \frac{2\delta_n^2}{\alpha_n^2 - 1}\int\limits_1^\infty \left\{\frac{\alpha_n J_0(k\beta x) - J_0(\beta x)}{x^2 - \delta_n^2}\right\}^2 \frac{x\,dx}{\sqrt{x^2-1}}. \tag{3.1}$$

An integral of infinite limits $(1,\infty)$ integral of finite limits $(0,\pi/2)$ have been transformed to

$$\chi_n = \frac{2\delta_n^2}{\alpha_n^2 - 1}\int\limits_0^{\pi/2} \left\{\frac{\alpha_n J_0(k\beta/\sin u) - J_0(\beta/\sin u)}{1 - \delta_n^2\sin^2 u}\right\}^2 \sin^2 u\, du \tag{3.2}$$

which is adequate formula for reactance for numerical calculations.

Integrating in the formula (3.1) we achieve with assumption that interference parameter $\beta = k_0 r_1 \gg 1$. Asymptotic formulas have been used

$$\begin{aligned} J_0^2(\beta x) &\sim \frac{1+\sin 2\beta x}{\pi \beta x}, \\ J_0(k\beta x) J_0(\beta x) &\sim \frac{1}{\pi \beta x \sqrt{k}} \left[\cos(k-1)\beta x + \sin(k+1)\beta x\right] \end{aligned} \tag{3.3}$$

and

$$\int_1^\infty \frac{e^{imx}\, dx}{\sqrt{x^2-1}\,(x^2-\delta_n^2)^2} = \sqrt{\frac{\pi}{2m}} \left\{ (1-\delta_n^2)^{-2} + O\left(\frac{1}{m}\right) \right\} e^{i(m+\pi/4)} \tag{3.4}$$

if $m \gg 1$.

Also an integral formula is helpful

$$\int_1^\infty \frac{dx}{\sqrt{x^2-1}\,(x^2-\delta_n^2)^2} = \frac{1}{2\delta_n^2\,(1-\delta_n^2)} \left(1 + \frac{2\delta_n^2 - 1}{\delta_n \sqrt{1-\delta_n^2}} \arcsin \delta_n \right). \tag{3.5}$$

After using the asymptotic method of calculations instead the formula (3.1) we get

$$\begin{aligned} \chi_n = \left[\pi\beta(\alpha_n^2-1)(1-\delta_n^2)\right]^{-1} &\left[\left(1+\frac{\alpha_n^2}{k}\right) \left(1 + \frac{2\delta_n^2-1}{\delta_n\sqrt{1-\delta_n^2}} \arcsin \delta_n \right) \right. \\ &\left. + \sqrt{\pi/\beta}\, \frac{\delta_n^2}{1-\delta_n^2} F_n(k,\beta) \right] \end{aligned} \tag{3.6}$$

with error $O(\delta_n^2\, \beta^{-3/2})$, where the function

$$\begin{aligned} F_n(k,\beta) \equiv \sin\left(2\beta+\frac{\pi}{4}\right) + \frac{\alpha_n^2}{k\sqrt{k}} \sin\left(2k\beta + \frac{\pi}{4}\right) \\ - 2\sqrt{2}\frac{\alpha_n}{\sqrt{k}} \left[\frac{\sin\left((k+1)\beta+\frac{\pi}{4}\right)}{\sqrt{k+1}} + \frac{\cos\left((k-1)\beta+\frac{\pi}{4}\right)}{\sqrt{k-1}} \right] \end{aligned} \tag{3.7}$$

has essential influence on "oscillating" character of reactance (3.6). The formula (3.6) represents the elementary form of an expression, convenient for calculations of normalized reactance of an annular membrane in the case of high frequency of radiated waves, when the source vibrates with n-th axially-symmetric mode.

4. Concluding remarks

As a result of theoretical analysis of the planar, annular membrane radiation formulas (3.1) and (3.6) for normalized radiation reactance have been derived.

Elementary formula (3.6) can be used for numerical calculations only if condition $x_n < \beta = k_0 r_1$ is satisfied. There have been isolated components, which have essential influence on "oscillating" character of changes of radiation reactance (Fig. 1, 2 and 3).

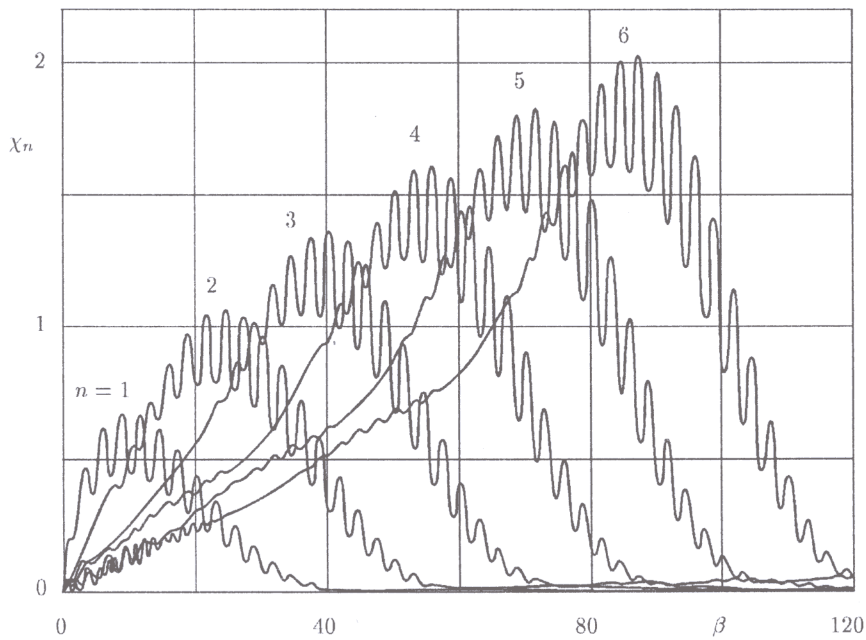

Fig. 1. Normalized reactance of radiation of a planar annular membrane versus β for the modes $(0, n)$, $n = 1, 2, \ldots, 6$ and $k = 1.2$.

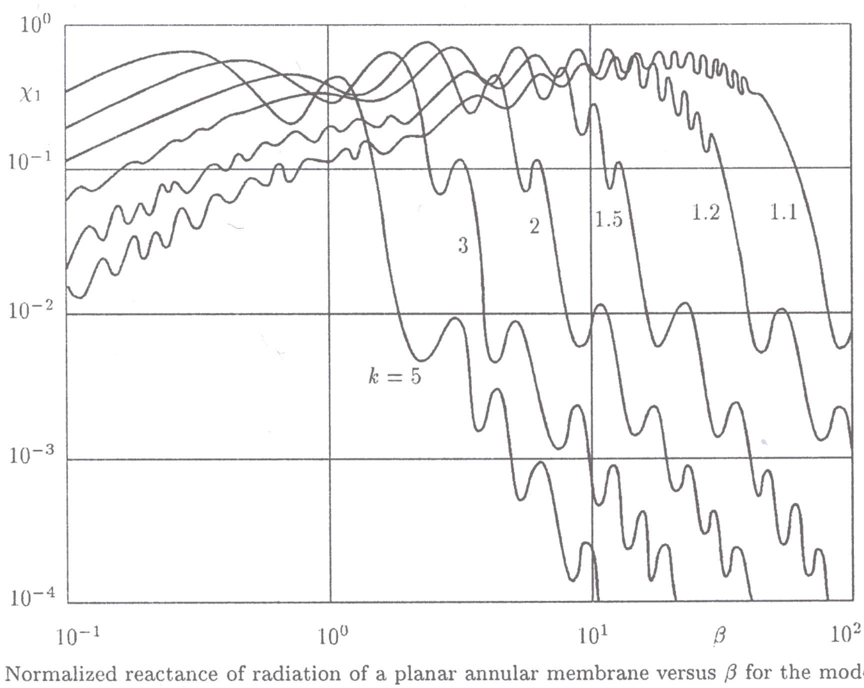

Fig. 2. Normalized reactance of radiation of a planar annular membrane versus β for the mode $(0, 1)$ and different k.

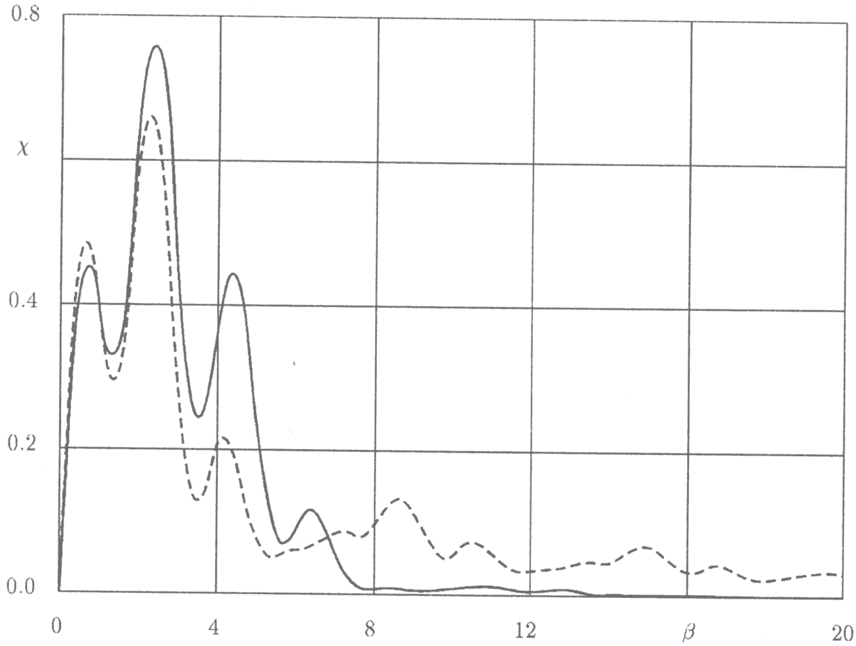

Fig. 3. Normalized reactance of radiation of a planar annular source versus parameter β for $k = 2$. The annular membrane for the mode $(0, 1)$ – solid line, the annular piston – dashed line.

In the case when the condition $x_n < \beta$ is not satisfied or when high precision of results is necessary, calculations must be made using the integral formula (3.1).

For comparison — apart of frequency characteristics of vibrating annular membrane — there have been presented a graph of radiation reactance of an annular piston.

On the basis of presented graphical illustration one can draw a conclusion that the radiation reactance of an annular membrane depends on: its sizes $k = r_2/r_1$, its form of vibrations $(0, n)$ and the interference parameter $\beta = k_0 r_1$.

Obtained formulas (3.1) and (3.6) can be used to describe the acoustic radiation of a vibrating system more complicated and less idealized. The example of analysis of such a system can be calculation of distribution of acoustic pressure or acoustic power, radiated by an annular membrane with consideration of the vibration force factor. Taking into account the reflexive interaction of radiated pressure wave of surrounding medium is also possible.

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