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THE INFLUENCE OF ACOUSTIC NONLINEARITY ON ABSORPTION PROPERTIES OF HELMHOLTZ RESONATORS PART I. THEORY

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This is the first of two companion papers concerned with the nonlinear absorption of Helmholtz resonators at a high amplitude incident wave. The phenomenon has been examined theoretically by use of a model of the acoustic field in the neighbourhood of the resonator placed at the end of cylindrical tube. The calculation results have shown that the peak of the absorption coefficient occurs when the nonlinear resistance is equal to the radiation resistance of the resonator. The full experimental investigations of this phenomenon and a comparison between theoretical and experimental data will be presented in the companion paper (Part II).

1. Introduction

The nonlinear properties of Helmholtz resonators occur due to the dependence of the resonator resistance on the particle velocity in the orifice. In the case of large amplitudes of velocity a part of acoustic energy is lost on account of turbulent motion nearby the edge of the orifice. The transfer of acoustic energy to the nonacoustic kinetic energy of these turbulences influences the variability of the absorption coefficient of resonator in the function of the sound intensity.

Nonlinear studies of Helmholtz resonators are scarce in number and relatively recent. Important among these are the experimental investigations by INGARD [1], BIES and WILSON [2], CZARNECKI [3, 4], WU and RUDNICK [5], as well as the theoretical work of ZINN [6]. One of the important results of these investigations is the observation that the resistance of a Helmholtz resonator increases with growing an amplitude of incident pressure. CZARNECKI [3] investigated an influence of the nonlinear properties of Helmholtz resonators on acoustic conditions in enclosures. He found an increase or a decrease of the absorption coefficients depending on the conditions of the resonator surroundings. WU and RUDNICK [5] measured a variation in the resonant frequency at high sound intensities. They observed shifts of the tuning curves of resonators towards higher frequencies with increasing amplitude of sound pressure. They ascribed these shifts to decrease of the end correction of the resonators at higher sound pressure.

The aim of this first of the two companion papers is to offer a theoretical model of the energy absorption mechanism, that occurs when a single Helmholtz resonator is excited by high amplitude plane wave. In the analysis it is considered the case of low frequency incident wave. The theoretical description of the acoustic field is based on the momentum equation for incompressible fluid in the space with rotating fluid motion and Bernoulli's equation in the space where fluid motion is irrotational. In this manner, the assumption used in earlier theoretical studies of acoustic nonlinearity [7, 8] that the fluid motion might be treated as being irrotational is removed. A loss resistance derived from the theoretical analysis is included in an impedance model of resonator to explain a change in absorption coefficient of the resonator. In the second paper [9] the comparison between experimental results and theoretical data will be presented.

2. Nonlinear effect at high amplitude incident sound

The present theoretical study is concerned with the interaction between high amplitude sound wave and a Helmholtz resonator placed at the end of cylindrical tube. The resonator consists of a part of this tube, and at one end it is terminated by rigid wall, while at the other one by a rigid plate with a centrally located circular orifice. The resonator dimensions are considerably smaller than the acoustic wavelength. When the amplitude of incident wave is large, the instantaneous flow pattern is different on both sides of the resonator orifice. In the first place, attention is focused on a flow structure during the first half of the cycle, when the flow is directed from tube to the resonator cavity (Fig. 1). Inside the tube, in the acoustic far field, the streamlines are parallel because only plane waves propagate in the area lying a long distance from the resonator. At the inflow side of the orifice the streamlines converge producing an acoustic near field, in which a reactive part of acoustic energy is concentrated. At the high amplitude of incident sound and a small ratio between orifice and tube diameters, there is a strong acoustic flow through the orifice which results in the separation of boundary layer and the formation of high velocity axial jet. When the edges of the orifice are sharp, the streamlines in the jet somewhat converge forming so-called *vena contracta*. The viscous interaction of the jet with the quiescent surroundings results in the formation of vortex ring that moves away from the orifice and dissipates into turbulence. During the second half of the cycle the flow in the resonator orifice reverses direction and the formation of the vortex ring occurs inside the tube.

To proceed with theoretical development the following assumptions will be required: (I) the frequency of incident wave is low; under this condition the fluid in the neighbourhood of the orifice may be treated as incompressible, (II) viscous forces are small compared with inertia forces, (III) the formation and the dissipation of the vortex ring occur in the small distance from the orifice, (IV) the flow pattern at the outflow side of the orifice is symmetrical during the cycle.

First we will consider the situation when the acoustic flow is directed from the tube to the resonator cavity (Fig. 1). The assumption, that a fluid motion between cross-sections 1 and 4 behaves as if it were incompressible and nonviscous, yields the following mo-

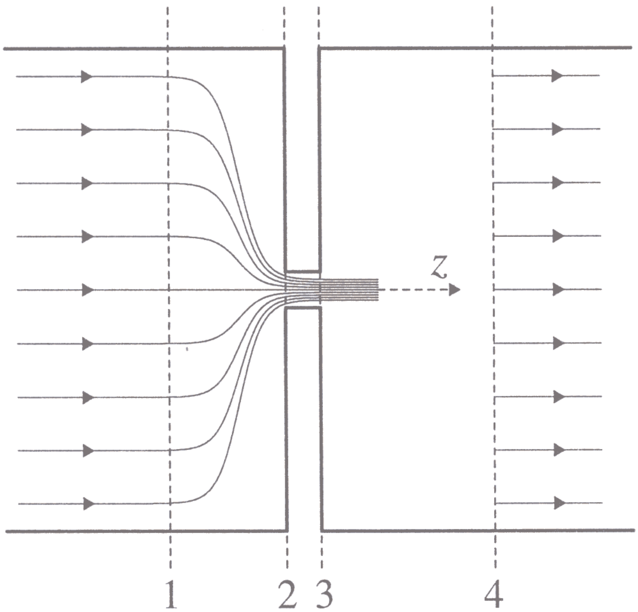

Fig. 1. Streamlines in the neighbourhood of the resonator orifice at high amplitude incident plane wave.

mentum equation

$$\varrho\frac{\partial \mathbf{U}}{\partial t} + \varrho(\mathbf{U}\cdot\nabla)\mathbf{U} + \mathrm{grad}(P) = 0. \tag{1}$$

In the above equation ϱ, $\mathbf{U}$ and P respectively denote the density of fluid, the velocity vector and the total pressure, $P = p + p_0$, where p_0 represents the equilibrium pressure. The velocity $\mathbf{U}$ is a superposition of the acoustic velocity $\mathbf{u}$ and the velocity $\mathbf{v}$ induced by the vortex ring which was formed at the outflow side of the orifice. At the inflow side, in the area between cross-sections 1 and 3, the fluid motion is irrotational ($\mathbf{v} = 0$), then the momentum equation (1) may be reduced to Bernoulli's equation

$$\varrho\frac{\partial \psi}{\partial t} + \frac{\varrho \mathbf{u}^2}{2} + P = \mathrm{const}, \tag{2}$$

which describes the relation between the pressure, the acoustic velocity and the velocity potential ψ along the streamline defined by $\mathbf{u}$. Since the streamlines in the cross-sections 1 and 3 are parallel then the pressures P_1, P_3 and the velocities u_1, u_3 in the planes 1 and 3 are uniform. In this case the mass conservation law gives

$$S_1 u_1 = S_0 u_0\,, \qquad u_3 = u_0/C_c\,, \tag{3}$$

where $S_1 = \pi a^2$ and $S_0 = \pi b^2$ respectively denote the cross-sectional area of the tube and the orifice area, C_c is a contraction coefficient and u_0 is the average velocity in the orifice area

$$u_0(t) = \frac{1}{S_0}\int_0^{2\pi}\int_0^{b} \mathbf{u}_2\cdot\mathbf{n}_z r\,dr\,d\phi, \tag{4}$$

where $\mathbf{u}_2$ is the velocity in the cross-section 2 and $\mathbf{n}_z$ is the unit vector in a direction of the z axis. The coefficient C_c is an experimentally determined quantity, and it is known to be quite sensitive to orifice shape. In the case of sharp-edged orifices C_c approximately equals 0.61 [10]. A combination of Eqs. (2) and (3) gives

$$p_1 + p_0 - P_3 = -\varrho \frac{\partial}{\partial t}(\psi_1 - \psi_3) + \frac{\varrho u_0^2}{2}\left(\frac{1}{C_c^2} - \frac{S_0^2}{S_1^2}\right). \tag{5}$$

A difference between potentials ψ_1 and ψ_3 has the same value along each streamline. If we consider the streamline along the axis of the orifice, we have $\psi = \int u_z dz$, where $u_z(z,t)$ is the velocity in the direction of z axis. The equation (5) can thus be written in the form

$$p_1 + p_0 - P_3 = \varrho \frac{\partial u_0}{\partial t} \int_1^3 F(z)\, dz + \frac{\varrho u_0^2}{2}\left(\frac{1}{C_c^2} - \frac{S_0^2}{S_1^2}\right), \tag{6}$$

where $F(z) = u_z/u_0$. The first term on the right-hand side of Eq. (6) is the pressure drop caused by an energy concentration in the reactive acoustic field. The integral in this term represents the effective orifice thickness being a sum of two parts

$$\int_1^3 F(z)\, dz = w_c d + \int_1^2 F(z)\, dz = w_c d + \Delta d_o\,, \tag{7}$$

where w_c is the correction factor of orifice thickness d, $1 < w_c < 1/C_c$, and Δd_o is the outside end correction which results from the convergence of streamlines on the external side of the resonator orifice. It is reasonably to expect that Δd_o is very close to the end correction at a small amplitude incident wave (linear case). Thus, the analytical determination of this quantity requires a more detailed analysis of acoustic field in the vicinity of the orifice.

Inside the resonator cavity, in the volume V bounded by the cross-sections 3 and 4 (Fig. 1), the fluid motion is rotational due to the formation of vortex ring. The total velocity $\mathbf{U}$ is thus a sum of the acoustic velocity $\mathbf{u}$ and the velocity $\mathbf{v}$ induced by the vortex. In the plane 3 the velocity satisfies the condition of continuity $u_3 = U_3$, while the pressure is uniformly distributed because outside the orifice, on the rigid wall, the pressure is approximately the same as in the jet [11]. In the plane 4, where the velocity $\mathbf{v}$ vanishes, the acoustic velocity is uniform and has the value $u_4 = u_0 S_0/S_1$. Since in the volume V the condition $\operatorname{div}\mathbf{U} = 0$ is satisfied then taking into account the identity [12]

$$\int_V [(\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{b}(\operatorname{div}\mathbf{a})]dv = \oint_S \mathbf{b}(\mathbf{a}\cdot\mathbf{n}_s)ds, \tag{8}$$

which is true for any vectors $\mathbf{a}$ and $\mathbf{b}$, it is possible to perform the momentum equation (1) into the following integral form

$$\varrho \int_V \frac{\partial \mathbf{U}}{\partial t} dv + \oint_S [\varrho \mathbf{U}(\mathbf{U}\cdot\mathbf{n}_s) + P\mathbf{n}_s]\, ds = 0, \tag{9}$$

where S is a boundary of the volume V and $\mathbf{n}_s$ is a unit vector directed away from the surface S. Because a normal component of velocity vanishes on rigid walls $(\mathbf{U}\cdot\mathbf{n}_s = 0)$ and a pressure distribution in the volume V is axially symmetrical then from Eq. (9) one can obtain

$$P_3 - p_4 - p_0 = \frac{\varrho}{S_1}\frac{\partial}{\partial t}\int_V (\mathbf{U}\cdot\mathbf{n}_z)dv + \varrho u_0^2\left(\frac{S_0^2}{S_1^2} - \frac{S_0}{S_1 C_c}\right). \tag{10}$$

Using the mass conservation law it can be shown that

$$\int_V (\mathbf{U}\cdot\mathbf{n}_z)dv = \int_3^4 \left[\int_0^{2\pi}\int_0^a (\mathbf{U}\cdot\mathbf{n}_z)r\,dr\,d\phi\right] dz = S_0 u_0 \int_3^4 dz = S_0 u_0 \Delta l\,, \tag{11}$$

where Δl is a distance between cross-sections 3 and 4. Substituting Eq. (11) into Eq. (10) and using Eq. (6) to eliminate the pressure P_3 yields the following equation that determines the pressure drop $p_1 - p_4$ during the first half of the cycle

$$p_1 - p_4 = \varrho\frac{\partial u_0}{\partial t}(w_c d + \Delta d_o + S_0\Delta l/S_1) + \frac{\varrho u_0^2}{2}\left(\frac{1}{C_c} - \frac{S_0}{S_1}\right)^2. \tag{12}$$

Taking into account the assumption (IV) it becomes easy to proceed with the theoretical analysis during the second half of the cycle, when the flow is directed from the resonator cavity to the tube. Because of a different shape of acoustic near field on both sides of the orifice, the considerations analogous as made above give the following equation

$$p_1 - p_4 = \varrho\frac{\partial u_0}{\partial t}(w_c d + \Delta d_i + S_0\Delta l/S_1) - \frac{\varrho u_0^2}{2}\left(\frac{1}{C_c} - \frac{S_0}{S_1}\right)^2, \tag{13}$$

where Δd_i is the inside end correction. The opposite sign in front of the term proportional to u_0^2 is a main difference between Eqs. (12) and (13) because the outside and inside end corrections, as will be shown in Sec. 3, considerably differ only at very small ratios l/a, where l and a are the resonator length and the cavity radius. These end corrections are, however, nearly equal for a typical resonator geometry, in which l/a is usually not too small $(l/a \geq 1/2)$. In this case Eqs. (12) and (13) may be written as one equation

$$p_1 - p_4 = \varrho\frac{\partial u_0}{\partial t}(d + \Delta d_n) \pm \frac{\varrho u_0^2}{2}\left(\frac{1}{C_c} - \frac{S_0}{S_1}\right)^2, \tag{14}$$

that applies throughout the duration of the whole cycle. In the above equation Δd_n, on the analogy of linear case, denotes the total end correction at high amplitude

$$\Delta d_n = (w_c - 1)d + \frac{1}{2}\Delta d + S_0\Delta l/S_1\,, \tag{15}$$

where $\Delta d = \Delta d_i + \Delta d_o$ is the total end correction at small amplitude incident sound. In Eq. (14) the pressure drop defined by the first term on the right-hand side is associated with an acoustic reactance. It is a part of the total reactance of resonator because it only describes a mass inertia in the orifice and the near acoustic field. The second term on the right-hand side, which not appears in the linear case, is a resistive term associated with losses due to the conversion of acoustic energy into vortical energy. Since this term is proportional to the square of mean orifice velocity, it represents a nonlinear part of acoustic pressure.

It follows from Eq. (14), that in a range of high amplitude of the orifice velocity the resistive term is much larger than the reactive one. Now, if one suppose that the pressure drop $\Delta p = p_1 - p_4$ is a harmonic function of the time, $\Delta p \sim \cos(\omega t)$, then from Eq. (14) one can conclude that

$$u_0(t) \approx \pm U_0 |\cos(\omega t)|^{1/2}, \tag{16}$$

where U_0 denotes the peak value of orifice velocity and the plus sign holds when $\cos(\omega t) \geq 0$, while the minus sign when $\cos(\omega t) < 0$. The expression for u_0 may be expand in a Fourier series and according to INGARD [13]

$$\begin{aligned} u_0(t) &= U_0\sqrt{\pi/2}\sum_{n=0}^{\infty}[\Gamma(7/4+n)\Gamma(3/4-n)]^{-1}\cos[(2n+1)\omega t] \\ &= U_0[1.11\cos(\omega t) - 0.159\cos(3\omega t) + 0.072\cos(5\omega t) - 0.043\cos(7\omega t) + \ldots], \end{aligned} \tag{17}$$

where Γ is the Gamma function. Thus, due to the nonlinearity the orifice velocity will be distorted so that its frequency spectrum will contain harmonic components.

3. Impedance of resonator

The losses resulting from the absorption of acoustic energy by vortical field may be included into a impedance model of the resonator by insertion of an additional orifice resistance. Since in Eq. (17) the first term in the Fourier series is much larger than the next ones, then it is possible to define this resistance in terms of a fundamental component of the orifice velocity

$$v_0(t) = 1.11\, U_0 \cos(\omega t) = V_0 \cos(\omega t). \tag{18}$$

Thus, using the approximation $u_0 \approx v_0$ in the reactive term of Eq. (14) and Eq. (16) in the resistive term yields the following expression for the pressure drop

$$p_1 - p_4 = \varrho \frac{\partial v_0}{\partial t}(d + \Delta d_n) + \pi b^2 R_n v_0 , \tag{19}$$

where R_n is a loss resistance in the case of nonlinearity

$$R_n = \frac{\varrho V_0}{2.46\,\pi b^2}\left(\frac{1}{C_c} - \frac{S_0}{S_1}\right)^2 . \tag{20}$$

Losses of acoustic energy at high sound intensity are then proportional to the amplitude of fundamental component of the orifice velocity. Since in this case there is a square-law relation between the pressure and the velocity amplitude, R_n is so-called the nonlinear orifice resistance (at the low intensity of incident sound the relation between the pressure and the velocity amplitude is linear). It is important to note that due to the form of

Eq. (19) it is now possible to replace the orifice velocity v_0 by its Fourier representation $\hat{v}_0 = V_0 e^{-j\omega t}$. Thus, the equation equivalent to Eq. (19) is following

$$\frac{\hat{p}_1 - \hat{p}_4}{\pi b^2 \hat{v}_0} = R_n - jk\frac{\varrho c}{\pi b^2}(d + \Delta d_n), \tag{21}$$

where $\hat{p}_1$ and $\hat{p}_4$ denote complex pressures. In order to find a dependence between the orifice velocity and the incident sound pressure it is necessary to determine the pressures $\hat{p}_1$ and $\hat{p}_4$. In the cross-section 1 a pressure distribution is uniform because the streamlines are parallel (Fig. 1). Thus, $\hat{p}_1$ is the pressure in a acoustic far field being a superposition of plane incident and reflected waves, including the plane wave radiated by resonator.

The pressure $\hat{p}_t$ inside the tube in the far field area can be determined by use of a classic theory of sound radiation. According to this theory, the response of resonator under external excitation may be analyzed as a sound production by a vibrating rigid piston located in the resonator orifice [14]. If the origin of a cylindrical coordinate system (r, ϕ, z) lies at the center of the orifice, and in addition the plane $z = 0$ covers the left-hand side of it, then for incident pressure $\hat{p}_i = \widehat{P}_i e^{j(kz-\omega t)}$ a formula describing $\hat{p}_t$ can be written as

$$\hat{p}_t(z,t) = 2\widehat{P}_i \cos(kz) e^{-j\omega t} - \varrho \frac{\partial}{\partial t} \int_0^{2\pi} \int_0^{b} \hat{v}_0(t)\, g_t(z_0 = 0)\, r_0\, dr_0\, d\phi_0\,, \tag{22}$$

where $g_t(z, z_0)$ represents Green's function for the plane wave motion inside the tube and (r_0, ϕ_0, z_0) is a position of source point. The first term on the right-hand side of Eq. (22) is a sum of incident and reflected waves. The quantity g_t can be simply obtained from the expression for the general Green's function G_t which includes both acoustic far and near field components. The function G_t is a solution of the wave equation and satisfies the boundary conditions

$$\frac{\partial G_t}{\partial r}(r = a) = \frac{\partial G_t}{\partial z_0}(z_0 = 0) = 0, \tag{23}$$

then for waves starting inside the tube from $z = -\infty$ is given by [15]

$$G_t = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} g_{mn} \cos(k_{mn} z_0) \exp(-jk_{mn} z), \tag{24}$$

where

$$g_{mn} = \frac{j}{\pi a^2} \epsilon_m \cos[m(\phi - \phi_0)] \frac{J_m(\gamma_{mn} r/a) J_m(\gamma_{mn} r_0/a)}{k_{mn}[1 - (m/\gamma_{mn})^2] J_m^2(\gamma_{mn})} \tag{25}$$

and γ_{mn} is the nth root of the equation $dJ_m(\gamma)/d\gamma = 0$, $k_{mn} = [k^2 - (\gamma_{mn}/a)^2]^{1/2}$, and ϵ_m is the Neumann factor, $\epsilon_0 = 1$, $\epsilon_m = 2$ $(m > 0)$. As may be seen, the function g_t is determined by the first term in the series on the right-hand side of Eq. (24) which is independent on r and r_0. Following from this, one can write Eq. (22) as

$$\hat{p}_t(z,t) = 2\widehat{P}_i \cos(kz) e^{-j\omega t} - \varrho c\, (b/a)^2 V_0 e^{-j(kz+\omega t)}, \tag{26}$$

where the second term on the right-hand side represents the plane wave radiated by resonator. Since it was assumed that a frequency of the incident wave is small, then a distance between the cross-section 1 and the orifice plane is much less than the wavelength. Under this condition from Eq. (26) one can easily obtain an expression for the unknown pressure $\hat{p}_1$

$$\hat{p}_1(t) = \hat{p}_t(z,t)_{|kz\to 0} = \left[2\widehat{P}_i - \varrho c\,(b/a)^2 V_0\right] e^{-j\omega t}. \tag{27}$$

The same method, as presented above, will be used for a determination of the pressure $\hat{p}_4$. In order to simplify the analysis we move the origin of a coordinate system to a plane which covers the right-hand side of the orifice (Fig. 1). In a small distance from the orifice plane a pressure in the resonator cavity is uniform, because it represents a superposition of multiple plane wave reflections. The formula for this pressure is thus given by

$$\hat{p}_c(z,t) = \varrho\frac{\partial}{\partial t}\int\limits_0^{2\pi}\int\limits_0^{b} \hat{v}_0(t)\, g_c(z_0=0)\, r_0 dr_0 d\phi_0\,, \tag{28}$$

where $g_c(z,z_0)$ is the Green's function for the plane wave motion inside the resonator cavity. It is a part of a general Green's function G_c derived for the resonator interior. Since G_c must satisfy boundary conditions

$$\frac{\partial G_c}{\partial r}(r=a) = \frac{\partial G_c}{\partial z_0}(z_0=0) = \frac{\partial G_c}{\partial z}(z=l) = 0, \tag{29}$$

then it may be expressed as follow

$$G_c = j\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{mn}\cos(k_{mn}z_0)\left[\sin(k_{mn}z) + \cos(k_{mn}z)\cot(k_{mn}l)\right]. \tag{30}$$

Since the function g_c is determined by the first term of this series then taking into account Eq. (28) one can write the expression for the pressure $\hat{p}_4$ as

$$\hat{p}_4(t) = \hat{p}_c(z,t)_{|kz\to 0} = j\varrho c\,(b/a)^2 V_0 \cot(kl) e^{-j\omega t}. \tag{31}$$

Finally, substituting Eqs. (26) and (31) into Eq. (21) yields the following formula for the acoustic impedance of resonator

$$Z = \frac{2\widehat{P}_i}{\pi b^2 V_0} = R_r + R_n + jX, \tag{32}$$

where $R_r = \varrho c/(\pi a^2)$ is the radiation resistance and X is the reactance of resonator

$$X = \frac{\varrho c}{\pi a^2}\cot(kl) - k\frac{\varrho c}{\pi b^2}(d + \Delta d_n). \tag{33}$$

The end correction $\Delta d = \Delta d_o + \Delta d_i$ which appears in a definition of Δd_n [Eq. (15)] represents the added mass effect at small amplitudes of incident sound. Thus, the quantity Δd may be determined by calculating the co-vibrating masses m_o and m_i on both sides of the resonator orifice. They are simply a product of air density and a result of double surface integration of these parts of Green's functions G_t and G_c which depend on radial coordinates

$$\begin{aligned} m_o &= \varrho\pi b^2 \Delta d_o = \varrho \int_{S_0}\int_{S_0} (G_t - g_t)_{|z,z_0=0}\, r_0 r\, dr_0 dr\, d\phi_0 d\phi, \\ m_i &= \varrho\pi b^2 \Delta d_i = \varrho \int_{S_0}\int_{S_0} (G_c - g_c)_{|z,z_0=0}\, r_0 r\, dr_0 dr\, d\phi_0 d\phi. \end{aligned} \tag{34}$$

After integration one can obtain expressions for outside and inside end corrections

$$\Delta d_o = \sum_{n=1}^{\infty} \frac{4aJ_1^2(\gamma_{0n} b/a)}{\gamma_{0n}^3 J_0^2(\gamma_{0n})}, \qquad \Delta d_i = \sum_{n=1}^{\infty} \frac{4aJ_1^2(\gamma_{0n} b/a)}{\gamma_{0n}^3 J_0^2(\gamma_{0n})} \coth\left(\frac{\gamma_{0n} l}{a}\right). \tag{35}$$

In the work of INGARD [1] it was assumed that for the cylindrical resonator placed at the end of a tube the outside end correction equals the inside one. The formula obtained for outside end correction agrees with Ingard's result. However, the exact expression for inside end correction derived in this paper indicates that Δd_o and Δd_i differ and it results from the additional factor $\coth(\gamma_{0n} l/a)$ in each term of the series. Since the quantity γ_{0n} is the nth root of the equation $dJ_0(\gamma)/d\gamma = 0$, then one has: $\gamma_{01} = 3.83$, $\gamma_{02} = 7.02$, $\gamma_{03} = 10.17$ ... . Thus, a difference between Δd_o and Δd_i is large at very small values of l/a, but Ingard's approximation $\Delta d_o \approx \Delta d_i$ is valid for $l/a \geq 1/2$.

4. Absorption coefficient of resonator

Inside the tube a long distance from the orifice plane an acoustic field is determined by a sum of two plane waves which propagate in positive and negative z direction. The first one represents an incident wave, while the second is a superposition of reflected and radiated waves. To proceed with theoretical analysis it is now necessary to introduce a complex reflection coefficient $\hat{\beta}$ which is defined as the ratio between pressure amplitude of the wave travelling in negative z direction to that of the incident wave. Thus, Eq. (26) for the pressure in the far field one can rewrite in the form

$$\hat{p}_t(z,t) = \widehat{P}_t e^{-j\omega t} = \widehat{P}_i \left(e^{jkz} + \hat{\beta} e^{-jkz}\right) e^{-j\omega t}. \tag{36}$$

From Eqs. (26) and (32) one can obtain the expression for the reflection coefficient

$$\hat{\beta} = \beta e^{j\chi} = 1 - 2Ae^{-j\varphi}, \tag{37}$$

where β denotes a modulus of the reflection coefficient and

$$A = \frac{R_r}{\sqrt{(R_r + R_n)^2 + X^2}}, \qquad \varphi = \arctan\left(\frac{X}{R_r + R_n}\right), \qquad -\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}. \tag{38}$$

Equation (36) may now be used to express the modulus of a pressure in far field area in terms of coefficient $\hat{\beta}$

$$|\widehat{P}_t| = |\widehat{P}_i| \sqrt{1 + \beta^2 + 2\beta \cos(2kz - \chi)}. \tag{39}$$

From Eqs. (39) the modulus of $\hat{\beta}$ can be obtained

$$\beta = \frac{|\widehat{P}_t|_{\max} - |\widehat{P}_t|_{\min}}{|\widehat{P}_t|_{\max} + |\widehat{P}_t|_{\min}} = \sqrt{1 + 4A^2 - 4A\cos(\varphi)} \tag{40}$$

and from this, the expression for the energy absorption coefficient

$$\alpha = 1 - \beta^2 = 4A[\cos(\varphi) - A]. \tag{41}$$

At low pressure amplitudes the nonlinear component of resistance can be neglected ($R_n = 0$). In this case the expression (38) for the quantity A yields

$$A = \frac{1}{\sqrt{1 + \tan^2(\varphi)}} = \cos(\varphi), \tag{42}$$

and this means, according to Eq. (41), that the absorption coefficient α always equals zero. This result is in full agreement with the basis of linear acoustics for nonviscous fluids. Thus, the conclusion may be derived that at high amplitude the changes in the absorption coefficient at the resonant frequency result from an increase of the resistance R_n. In accordance with the foregoing theory the absorption coefficient in the case of resonance ($X = 0$),

$$\alpha = 4A(1 - A) = \frac{4R_n/R_r}{(1 + R_n/R_r)^2} \tag{43}$$

increases with the growth of the nonlinear resistance R_n as long as $R_n < R_r$. When the equality $R_n = R_r$ is reached the absorption coefficient equals unity. It follows from Eq. (43) that further increase in the resistance R_n makes a contribution to decrease in the coefficient α.

A value of α cannot be computed directly from Eq. (41) because the parameter A and phase φ are dependent of the velocity amplitude V_0 which is a solution of the equation

$$2|\widehat{P}_i| - \pi b^2 V_0 |R_r + R_n + jX| = 0. \tag{44}$$

In order to solve Eq. (44) it is necessary to apply a numerical procedure because nonlinear resistance R_n depends on the orifice velocity [see Eq. (20)]. A simple analytic solution may be obtained only in the case of resonance and the result is

$$\frac{R_n}{R_r} = \frac{1}{2}\sqrt{1 + \frac{3.26}{\varrho c^2}\left(\frac{S_1}{S_0 C_c} - 1\right)^2 |\widehat{P}_i|} - \frac{1}{2}. \tag{45}$$

Equation (43) together with this solution enables to determine a dependence of the absorption coefficient α at a resonant frequency on the incident pressure amplitude $|\widehat{P}_i|$. As may be noted, there is such a value of $|\widehat{P}_i|$ for which the absorption coefficient equals unity. This amplitude may be calculated directly from Eq. (45) putting R_n equal R_r. The result is

$$|\widehat{P}_i| = \frac{2.46\varrho c^2 (b/a)^4}{[1/C_c - (b/a)^2]^2}, \tag{46}$$

so, the pressure amplitude, at which the absorption coefficient is equal to unity, depends on the ratio b/a only. Since this dependence would be useful in the practical resonator design it is reasonably to replace the pressure amplitude $|\widehat{P}_i|$ by its level L_i expressed in decibels. In Fig. 2, the level L_i calculated from Eq. (46) is plotted against the ratio b/a in the range $0.02 \leq b/a \leq 0.5$. One immediately sees that at very small values of b/a the maximum sound absorption occurs for relatively low pressure levels of incident wave ($L_i \approx 60$ dB). With growing b/a the level L_i fast increases, reaching the value $L_i \approx 175$ dB for the ratio $b/a = 0.5$.

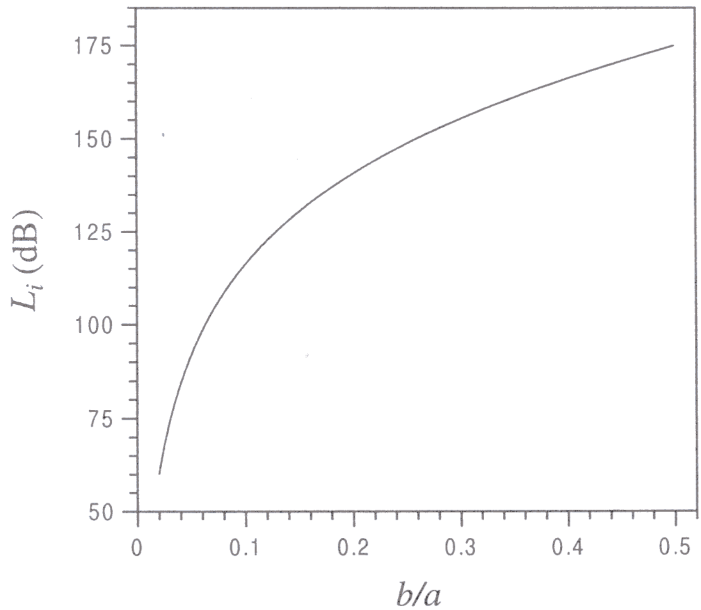

Fig. 2. Dependence of pressure level L_i on ratio b/a.

5. Discussion and conclusions

The acoustic nonlinearity phenomenon, which occurs when the Helmholtz resonator is excited by high amplitude sound wave, has been investigated in this paper. A theoretical model has been presented in order to explain an increase in acoustic energy losses with increasing sound intensity. It has been shown that the nonlinear loss mechanism is associated with a flow separation on the orifice edge and a formation of high speed jet on the outflow side of the orifice, because the interaction of jet with motionless medium results in a generation of the vortical field which extracts energy from acoustic field.

In the theoretical study a pressure field was determined separately in the area with irrotational and rotational fluid motion. It is important to emphasize that this theory may also be applied in the case of high amplitude acoustic transmission through an orifice plate in a pipe. As compared to the model of CUMMINGS [8], in which the rotational fluid motion was totally ignored, the present study gives a more complete description of the pressure field in the orifice surroundings.

It was found that a total pressure drop due to the jet formation and generation of vorticity is proportional to the square of the orifice velocity amplitude. This dependence for sinusoidal excitation signals leads to a harmonic distortion of the orifice velocity. As a result of this distortion the complications arise in definition of the resonator impedance. In the present study this problem was overcome by determining this impedance for the fundamental component of orifice velocity. If in the impedance model an acoustic energy loss due to a viscosity is neglected, then the resistive part of the resonator impedance consists of the radiation resistance and the nonlinear resistance. It has been shown that the absorption coefficient of resonator reaches the unity when the radiation resistance is equal to the nonlinear one and for a given resonator geometry it occurs at the exactly determined value of incident pressure amplitude.

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